MTH5105 Differential and Integral Analysis 2010-2011

Exercises 9

These exercises do *not* constitute coursework, but their content is definitely examinable. Model solutions will be made available on the course webpage by the last day of term. Starred questions are more difficult than unstarred ones.

Exercises

1) (a) Show that for all $x \in \mathbb{R}$, the sum $\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$ converges.
[You may use that $|\sin(t)| \leq |t|$ for all $t \in \mathbb{R}$.]

   (b) Show that the sum $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos\left(\frac{x}{k}\right)$ converges uniformly for all $x \in \mathbb{R}$.

   (c) Deduce that $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{x}{k}\right)$$

   is differentiable.

2) Is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sum\limits_{k=1}^{\infty} \sin^2(x/k)$ differentiable?

3) Let $f_n : [0,1] \mapsto \mathbb{R}$ be a sequence of differentiable functions, and let $f : [0,1] \mapsto \mathbb{R}$. Consider the statements

   (a) $f_n \to f$ pointwise,

   (b) $f_n \to f$ uniformly,

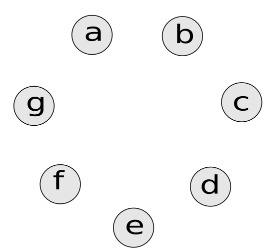
   (c) f_n' converges pointwise,

   (d) $f_n' \to f'$ pointwise,

   (e) f continuous,

   (f) f differentiable,

   (g) $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$,



   and cleary indicate in the enclosed figure all implications by the appropriate arrows ("$\Longrightarrow$").

*4) Let $f_n : [0,\infty) \mapsto \mathbb{R}$ be a sequence of continuous functions that converge uniformly to $f(x) = 0$. Show that if

$$0 \leq f_n(x) \leq e^{-x}$$

   for all $x \geq 0$ and for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,dx = 0\ .$$

   [Recall from Calculus I the definition of the improper integral $\int_0^{\infty} f(x)\,dx = \lim_{A\to\infty} \int_0^A f(x)\,dx$.]

Thomas Prellberg, March 2011