MATH 5105 Differential and Integral Analysis
Assignment 4 Solutions

(a) Let $g : [a,b] \to \mathbb{R}$ be bounded. We have proved that if g is Riemann integrable on $[a,b]$, then so is g^2. Prove or disprove the converse: if g^2 is Riemann integrable on $[a,b]$ then g is Riemann integrable on $[a,b]$.

The converse is false. Consider the function

$$g(x) = \begin{cases} 1 & x \text{ is rational and } x \in [a,b], \\ -1 & x \text{ is irrational and } x \in [a,b]. \end{cases}$$

For the function g, we note that every non-trivial subinterval of $[a,b]$ will contain both irrational and rational numbers. Therefore the maximum and minimum bounds of g, i.e. $M_i = 1$ and $m_i = -1$ are achieved in any subinterval of the partition $[x_{i-1}, x_i]$ of the partition $P = \{x_0, x_1, \ldots, x_n\}$. It follows that for *every* partition P of $[a,b]$, we have

$$U(g,P) = \sum_{i=1}^{i=n} M_i(x_i - x_{i-1}) = (1)\sum_{i=1}^{i=n}(x_i - x_{i-1}) = (x_n - x_0) = (b-a),$$

$$L(g,P) = \sum_{i=1}^{i=n} m_i(x_i - x_{i-1}) = (-1)\sum_{i=1}^{i=n}(x_i - x_{i-1}) = -(x_n - x_0) = -(b-a).$$

Therefore $U(g,P) - L(g,P) = (b-a) - (-(b-a) = 2(b-a)$, for *all* partitions P.

Thus the Riemann integrability condition

$$U(g,P) - L(g,P) < \epsilon$$

is not satisfied if $\epsilon < 2(b-a)$. Therefore g is not Riemann integrable on $[a,b]$.

By comparison, we have

$$g^2(x) = g(x) \cdot g(x = \begin{cases} 1 & x \text{ is rational and } x \in [a,b], \\ 1 & x \text{ is irrational and } x \in [a,b], \end{cases}$$

and we see that the function g^2 on $[a,b]$ satisfies $g^2(x) = 1$, for all $x \in [a,b]$, a constant function. Thus for any partition P of $[a,b]$, we have $M_i = m_i = 1$, and

$$U(g^2, P) = L(g^2, P) = \sum_{i=1}^{i=n} (1)(x_i - x_{i-1}) = (b-a),$$

and therefore, $U(g^2, P) - L(g^2, P) \equiv 0$ for any partition P. Therefore the function g^2 is Riemann integrable, with $\int_a^b g^2 = (b-a)$.

Thus, we see by counterexample that g^2 can be integrable when g is not integrable and the converse is not true.

(b) Assume that $h : \left[a, b^2\right] \to \mathbb{R}$ is a continuous function and let $G : [a,b] \to \mathbb{R}$ denote the following function,

$$G(x) = \int_a^{x^2} h(t)dt.$$

Show that G is differentiable and find its derivative.

Define $H(y) = \int_a^y h(t)dt$ for $y \in [a, b^2]$. By the fundamental theorem of calculus with h a continuous function, we have that H is differentiable with $H'(y) = h(y)$.

Now $G(x) = \int_a^{x^2} h(t)dt$, and so $G(x) = H(y)$ with $y = x^2$.

So we have

$$G'(x) = H'(y) \cdot y' = 2x \cdot h(x^2).$$

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