MTH5105 Differential and Integral Analysis 2012-2013

Solutions 7

1 Exercise for Feedback

1) (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\,dt$.
   (i) Why is f bounded?
   (ii) Prove that F is bounded.
   (iii) Prove that there exists a $c \in [a, b]$ such that $F(c) = \sup\{F(x) : x \in [a, b]\}$.
   (iv) Now suppose that f is continuous, and that the point c from (iii) satisfies $c \in (a, b)$ What can you conclude about $f(c)$?

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove or disprove: if f^2 (defined by $(f^2)(x) = f(x)^2$) is Riemann integrable on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Solution:

(a) (i) A Riemann integrable function must be bounded.
   (ii) By Theorem 8.4(a) we know that F is continuous on $[a, b]$, and hence bounded (by a result from Convergence & Continuity).
   An alternative proof is to note that

$$|F(x)| = \left|\int_a^x f(t)\,dt\right| \leq \int_a^x |f(t)|\,dt \leq (b - a)\sup\{f(t) : t \in [a, b]\}\,.$$

   (iii) By Theorem 8.4(a), F is continuous on $[a, b]$, hence (by a result from Convergence & Continuity) attains its upper bound for some $c \in [a, b]$
   (iv) We can conclude that $f(c) = 0$. Proof: By Theorem 8.4(b), if f is continuous then F is differentiable and $f(x) = F'(x)$. If F attains its maximum at $c \in (a, b)$, then by Theorem 2.1, $F'(c) = 0$. Hence $f(c) = 0$.

(b) This is false.
A counterexample is given by the bounded function

$$f(x) = \begin{cases} 1 & x \text{ rational,} \\ -1 & x \text{ irrational.} \end{cases}$$

Clearly $f^2(x) = 1$ (a constant function), and hence f^2 is integrable on $[a, b]$, but f is not (see the example in lectures of a non-integrable function, where we used values 0 and 1 instead of -1 and 1).

2 Extra Exercises

2) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that if $\int_a^b f(x)\,dx = 0$ then there exists a $c \in (a, b)$ such that $f(c) = 0$. *[Hint: use an antiderivative of f.]*

Solution:

We use

$$F(t) = \int_a^t f(x)\,dx \ .$$

Then $F(a) = 0$ and $F(b) = \int_a^b f(x)\,dx = 0$.

This should remind you of Rolle's Theorem. We need to check whether we can apply it:

As f is continuous, F is an antiderivative of f: it is differentiable on $[a,b]$ and its derivative $F' = f$ is continuous on $[a,b]$.

Thus the assumptions of Rolle's Theorem are satisfied, and we conclude that there is a $c \in (a,b)$ such that

$$0 = F'(c) = f(c) \ .$$

3) Compute $\lim_{n\to\infty} f_n(x)$ and $\lim_{n\to\infty} f_n'(x)$ for the following functions:

(a) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{\sin(nx)}{\sqrt{n}} \ .$$

(b) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{n}(\sqrt{1+n^2x^2} - 1) \ ,$$

(c) $f_n : \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \frac{1}{1+nx^2} \ .$$

If the limit doesn't exist, please indicate clearly for which values of x this is the case and give a brief indication why (no complete proof necessary).

Solution:

(a) $|f_n(x)| \le \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, hence

$$\lim_{n\to\infty} f_n(x) = 0 \ .$$

$f_n'(x) = \sqrt{n}\cos(nx)$. With increasing n, this function oscillates with strictly increasing amplitude and frequency, so

$$\lim_{n\to\infty} f_n'(x) \text{ does not exist.}$$

[A proof (not asked for) could be as follows. If $|\cos(nx)| \le 1/2$ then $|\cos(2nx)| \ge 1/2$. Thus, for all x there exists an increasing subsequence n_k such that $|\cos(n_k x)| \ge 1/2$. This implies $|f_{n_k}'(x)| \ge \sqrt{n}_k/2$, so $f_n'(x)$ cannot converge.]

(b) $f_n(x) = \sqrt{x^2 + 1/n^2} - 1/n$, hence

$$\lim_{n\to\infty} f_n(x) = |x| \ .$$

$f_n'(x) = nx/\sqrt{1+n^2x^2} = x/\sqrt{x^2+1/n^2}$, hence

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 1 & x > 0 \ , \\ 0 & x = 0 \ , \\ -1 & x < 0 \ . \end{cases}$$

(c) $f_n(x) = 1/(1+nx^2)$ so that $f_n(0) = 1$, and for $x \neq 0$ we have $|f_n(x)| < 1/(nx^2)$, hence

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 0\,, \\ 0 & x \neq 0\,. \end{cases}$$

$f_n'(x) = -2nx/(1+nx^2)^2$, so that $f_n'(0) = 0$, and for $x \neq 0$ we have $|f_n(x)| < 2/(n|x|^3)$, hence

$$\lim_{n\to\infty} f_n'(x) = 0\,.$$

4) For a bounded set $\Omega \subset \mathbb{R}$, show that $\sup_{y\in\Omega}|y| - \inf_{y\in\Omega}|y| \leq \sup_{y\in\Omega} y - \inf_{y\in\Omega} y$. *[This is needed in the proof of Theorem 7.7.]*

Solution:

This can be shown using a long chain of transformations:

$$\begin{aligned}
\sup_{y\in\Omega}|y| - \inf_{y\in\Omega}|y| &= \sup_{y\in\Omega}|y| - \inf_{x\in\Omega}|x| && \text{a change of variables}\\
&= \sup_{y\in\Omega}|y| + \sup_{x\in\Omega}(-|x|) && \text{change inf to sup}\\
&= \sup_{x,y\in\Omega}(|y| - |x|) && \text{combine terms}\\
&\leq \sup_{x,y\in\Omega}(|y - x|) && ||y| - |x|| < |y - x|\\
&= \sup_{x,y\in\Omega}(y - x) && \text{rhs is symmetric in } x \text{ and } y\\
&= \sup_{y\in\Omega} y + \sup_{x\in\Omega}(-x) && \text{split terms}\\
&= \sup_{y\in\Omega} y - \inf_{x\in\Omega} x && \text{change sup to inf}\\
&= \sup_{y\in\Omega} y - \inf_{y\in\Omega} y && \text{a change of variables}
\end{aligned}$$

*5) Evaluate

$$\lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin(nx)}{nx}\,dx\,.$$

Solution:

The strategy of the proof is to choose an $\epsilon > 0$ and consider the intervals $[0, \epsilon]$ and $[\epsilon, \pi/2]$ separately.

Using that $|\sin(t)| \leq |t|$, we estimate

$$\left|\int_0^{\epsilon} \frac{\sin(nx)}{nx}\,dx\right| \leq \int_0^{\epsilon} \left|\frac{\sin(nx)}{nx}\right| dx \leq \int_0^{\epsilon} dx = \epsilon\,.$$

Using that $|\sin(t)| \leq 1$, we estimate

$$\left|\int_{\epsilon}^{\pi/2} \frac{\sin(nx)}{nx}\,dx\right| \leq \int_{\epsilon}^{\pi/2} \left|\frac{\sin(nx)}{nx}\right| dx \leq \frac{1}{n}\int_{\epsilon}^{\pi/2} \frac{dx}{x} = \frac{1}{n}(\log(\pi/2) - \log\epsilon)\,.$$

Hence

$$\left|\int_0^{\pi/2} \frac{\sin(nx)}{nx}\,dx\right| \leq \epsilon + \frac{1}{n}(\log(\pi/2) - \log\epsilon)\,,$$

and choosing $\epsilon = 1/n$, we find

$$\left|\int_0^{\pi/2} \frac{\sin(nx)}{nx}\,dx\right| \leq \frac{1}{n}(1 + \log(\pi/2) + \log n) \to 0$$

as $n \to \infty$.