# MTH5105 Differential and Integral Analysis 2010-2011

Exercises 7

There are two sections. Questions in Section 1 will be used for feedback. Questions in Section 2 are voluntary but highly recommended. Starred questions are more difficult than unstarred ones.

## 1 Exercises for Feedback

1) (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\,dt$.
   (i) Why is $f$ bounded?
   (ii) Prove that $F$ is bounded.
   (iii) Prove that there exists a $c \in [a, b]$ such that $F(c) = \sup\{F(x) : x \in [a, b]\}$.
   (iv) Now suppose that $f$ is continuous, and that the point $c$ from (iii) satisfies $c \in (a, b)$ What can you conclude about $f(c)$?

   (b) Let $f : [a, b] \to \mathbb{R}$ be bounded. Prove or disprove: if $f^2$ is Riemann integrable on $[a, b]$ then $f$ is Riemann integrable on $[a, b]$.

## 2 Extra Exercises

2) Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that if $\int_a^b f(x)\,dx = 0$ then there exists a $c \in (a, b)$ such that $f(c) = 0$. *[Hint: use an antiderivative of $f$.]*

3) Compute $\lim_{n\to\infty} f_n(x)$ and $\lim_{n\to\infty} f_n'(x)$ for the following functions:

   (a) $f_n : \mathbb{R} \to \mathbb{R}$,
$$x \mapsto \frac{\sin(nx)}{\sqrt{n}} \; .$$

   (b) $f_n : \mathbb{R} \to \mathbb{R}$,
$$x \mapsto \frac{1}{n}(\sqrt{1 + n^2x^2} - 1) \; ,$$

   (c) $f_n : \mathbb{R} \to \mathbb{R}$,
$$x \mapsto \frac{1}{1 + nx^2} \; .$$

   If the limit doesn't exist, please indicate clearly for which values of $x$ this is the case and give a brief indication why (no complete proof necessary).

4) For a bounded set $\Omega \subset \mathbb{R}$, show that $\sup_{y\in\Omega} |y| - \inf_{y\in\Omega} |y| \le \sup_{y\in\Omega} y - \inf_{y\in\Omega} y$. *[This is needed in the proof of Theorem 7.7.]*

*5) Evaluate
$$\lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin(nx)}{nx}\,dx \; .$$

The deadline is 5.00pm (strict) on Monday 21st March. Please hand in your coursework to the orange coursework box on the second floor. Coursework will be returned during the exercise class immediately following the deadline.

Thomas Prellberg, March 2011