MATH 5105 Differential and Integral Analysis
Assignment 3 Solutions

1. Consider the function $u(y) = \frac{1}{y}$ on $[1, b]$, $b > 1$. Compute the lower and upper sum for the partition

$$P_n = \{y_0 = 1, y_1 = 1 + \frac{b-1}{n}, \cdots, y_k = 1 + \frac{k(b-1)}{n}, \cdots, y_n = b\}.$$

Using these two sums, show that u is integrable on $[1, b]$.

Proof. Note that u is a decreasing function. We take an equipartition $P_n = \{y_0 = 1, y_1 = 1 + \frac{b-1}{n}, \cdots, y_k = 1 + \frac{k(b-1)}{n}, \cdots, y_n = b\}$. We then compute that as $\Delta_i = y_i - y_{i-1} = \frac{b-1}{n}$ then

$$\begin{aligned} U(u, P_n) &= \sum_{i=1}^{n} \left(\sup_{[y_{i-1}, y_i]} u(y) \right) (y_i - y_{i-1}) \\ &= \frac{b-1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{(k-1)(b-1)}{n}} \end{aligned}$$

and

$$\begin{aligned} L(u, P_n) &= \sum_{i=1}^{n} \left(\inf_{[y_{i-1}, y_i]} u(y) \right) (y_i - y_{i-1}) \\ &= \frac{b-1}{n} \sum_{k=1}^{n} \frac{1}{1 + \frac{k(b-1)}{n}}. \end{aligned}$$

Then we see that

$$U(u, P_n) - L(u, P_n) = \frac{(b-1)^2}{bn}$$

so that

$$\lim_{n \to \infty} (U(u, P_n) - L(u, P_n)) = 0$$

and hence f is Riemann integrable. □