MATH 5105 Differential and Integral Analysis
Solutions to Assignment 1

1. Using the definition of continuity, show that the following functions are continuous

   (a)

$$q(z) = \frac{1}{z^3} \text{ at } z_0 \in (0, \infty).$$

Proof. Let $z_0 \in (0, \infty)$. We want to show that given an $\varepsilon > 0$, we can choose a $\delta = \delta(z_0, \varepsilon)$ such that if $|z - z_0| < \delta$ then

$$|g(z) - g(z_0)| < \varepsilon.$$

We therefore consider

$$|g(z) - g(z_0)| = \left| \frac{1}{z^3} - \frac{1}{z_0^3} \right| = \left| \frac{(z - z_0)(z^2 + zz_0 + z_0^2)}{z^3 z_0^3} \right|. \tag{0.1}$$

Since $z_0 \in (0, \infty)$, we know $z_0 > 0$. Therefore if we choose $\delta(z_0, \varepsilon) < \frac{|z_0|}{2}$ then by the reverse triangle inequality we have

$$|z| - |z_0|, |z_0| - |z| \leq |z - z_0| < \frac{|z_0|}{2}.$$

Rearranging, we find that $\frac{|z_0|}{2} < |z| < \frac{3|z_0|}{2} \implies \frac{1}{|z|} < \frac{2}{|z_0|}$. Inserting this into (0.1), we get

$$\begin{aligned} |g(z) - g(z_0)| \leq \left| \frac{(z - z_0)(z^2 + zz_0 + z_0^2)}{z^3 z_0^3} \right| &\leq \frac{8}{|z_0|^3 |z_0|^3} \left(\frac{9}{4}|z_0|^2 + \frac{3}{2}|z_0|^2 + |z_0|^2 \right) |z - z_0| \\ &= \frac{38}{|z_0|^4} |z - z_0|. \end{aligned}$$

Hence we choose $\delta(z_0, \varepsilon) = \frac{\varepsilon |z_0|^4}{38}$ we get

$$|g(z) - g(z_0)| = \left| \frac{(z - z_0)(z^2 + zz_0 + z_0^2)}{z^3 z_0^3} \right| \leq \frac{38}{|z_0|^4} |z - z_0| <= \varepsilon.$$

Recall that we made two choices $\delta(z_0, \varepsilon) < \frac{|z_0|}{2}$ and $\delta(z_0, \varepsilon) = \frac{\varepsilon |z_0|^4}{38}$ so finally we must choose

$$\delta(z_0, \varepsilon) = \min \left\{ \frac{|z_0|}{2}, \frac{\varepsilon |z_0|^4}{38} \right\}$$

□

2. Suppose that $g : I \to \mathbb{R}$ is differentiable at $x = x_0$. Prove the following limit exists

$$\lim_{h \to 0} \frac{g(x_0 + 6h) - g(x_0 - 6h)}{12h}.$$

Is the converse true? That is if the limit

$$\lim_{h \to 0} \frac{g(x_0 + 6h) - g(x_0 - 6h)}{12h} = L$$

exists is g differentiable at $x = a$? (Either prove your answer or give a counterexample).

(4 marks)

Proof. If g is differentiable at x_0 then

$$\lim_{h \to 0} \frac{g(x_0 + 6h) - g(x_0)}{6h} = g'(x_0)$$

and

$$\lim_{h \to 0} \frac{g(x_0 - 6h) - g(x_0)}{-6h} = g'(x_0)$$

Summing these two expressions we get

$$\lim_{h \to 0} \frac{g(x_0 + 6h) - g(x_0) - g(a - 6h) + g(x_0)}{6h} = 2g'(x_0)$$

or

$$\lim_{h \to 0} \frac{g(x_0 + 6h) - g(x_0 - 6h)}{12h} = g'(x_0).$$

The converse is not true - consider the function

$$g(x) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then computing we see that

$$\lim_{h \to 0} \frac{g(h) - g(-h)}{2h} = 1.$$

but the function is not even continuous at $x_0 = 0$ and hence is not differentiable. □