§9 Sequences & Series of Functions

WEEK 9
Lecture 22

§9.1 Sequences of Functions

Def^n 9.1.1. (Pointwise convergence)

A sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$, where $f_n: \Omega (\subseteq \mathbb{R}) \to \mathbb{R}$

is pointwise convergent to the function $f: \Omega \to \mathbb{R}$

if for each $x \in \Omega$, $f_n(x) \to f(x)$ as $n \to \infty$.

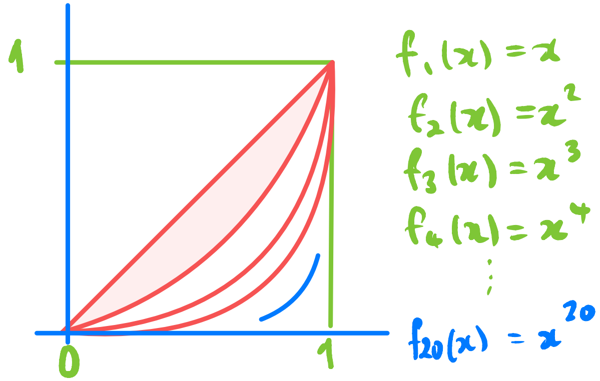
Note:

in terms of $\varepsilon - N$, fix $x_0 \in \Omega$ and let $a_n = f_n(x_0)$,

and $a = f(x_0)$, then $a_n \to a$ as $n \to \infty$.

Example 9.1.2.

(1) $f_n(x) = x^n, \quad 0 \le x \le 1.$



Note $x^n \to 0$ as $n \to \infty$, $\quad 0 \le x < 1.$

If $0 \le x < 1$, $\quad x = \dfrac{1}{1+y}$, $y > 0$ $\therefore$ $x^n = \dfrac{1}{(1+y)^n} = \dfrac{1}{\left(1 + ny + \frac{n(n-1)}{2!}y^2 + \dots\right)}$

$\Rightarrow \quad x^n < \dfrac{1}{ny}$ and $\dfrac{1}{ny} \to 0$ as $n \to \infty$

By sandwich theorem: $0 \le x^n < \dfrac{1}{ny}$ and $\displaystyle\lim_{n\to\infty} \frac{1}{ny} = 0$

$\therefore \displaystyle\lim_{n\to\infty} x^n = 0$, so $x^n \to f(x)$ where $0 \le x < 1$ $\therefore$ $f(x) \equiv 0$

However, for $x = 1$: $x^n \equiv 1$ $\therefore$ $x^n \to 1$ as $n \to \infty$ $\therefore$ $f(1) = 1$

Pointwise limit function $\quad f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$.

(ii) $f_n(x) = \frac{nx}{n+1}$; $f_n(x) \to x$ as $n \to \infty$. $\therefore f(x) = x$ pointwise limit

because $\frac{n}{n+1} \to 1$ as $n \to \infty$. $\therefore f(x) = x$, $\forall x \in \mathbb{R}$

Pointwise limit function

f_n. C

$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$ as $n \to \infty$.

(iii) $f_n(x) = \frac{1}{(1+x)^n}$, $x \geq 0$, $f(x) = ??$

Note $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

$\therefore nx < (1+x)^n$ for $x > 0$

and $\therefore \frac{1}{(1+x)^n} < \frac{1}{nx}$ and $\frac{1}{nx} \to 0$ as $n \to \infty$ $\quad x > 0$

$0 \leq \frac{1}{(1+x)^n} < \frac{1}{nx} \to 0$ $\therefore \frac{1}{(1+x)^n} \to 0$ for $x > 0$ $\therefore f(x) = 0$, $x > 0$ discont. $f(x)$.

$n \to \infty$

For $x = 0$: $f_n(0) \equiv 1 \Rightarrow f(0) = 1$

D.

In more usual notation:

$\forall \varepsilon > 0 \ \exists \ n \in N$, such that $\forall n > N$ $\quad a_n = f_n(x)$

$|a_n - a| < \varepsilon$

Note N may depend on both ε & x

So we note in the above examples:

pointwise limit fns can be continuous or $\left\{\begin{array}{l}\text{discontinuous} \\ \text{non-continuous}\end{array}\right.$!

Exercises (Ex 7)

(i) $f_n : \mathbb{R} \to \mathbb{R}$ $\qquad x \longmapsto \dfrac{\sin nx}{\sqrt{n}}$

$$0 \leq |f_n(x)| = \frac{|\sin nx|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \qquad \forall x \in \mathbb{R}$$

$$0 \leq \lim_{n\to\infty} |f_n(x)| \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \qquad \therefore \lim_{n\to\infty} |f_n(x)| = 0$$

$$\Rightarrow f(x) = 0, \quad \forall x \in \mathbb{R}.$$

(ii) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto \frac{1}{n}\left(\sqrt{1+n^2x^2} - 1\right)$

$a-b = \frac{a^2-b^2}{a+b}$

$a = \sqrt{1+n^2x^2}$
$b = 1$

Use $1+n^2x^2 - 1 = \left(\sqrt{1+n^2x^2}-1\right)\left(\sqrt{1+n^2x^2}+1\right)$

$$\frac{1}{n}\left(\sqrt{1+n^2x^2}-1\right) = \frac{1}{n}\frac{(n^2x^2)}{\sqrt{1+n^2x^2}+1} = \frac{nx^2}{\sqrt{1+n^2x^2}+1}$$

$$= \frac{nx^2}{n|x|\left(\sqrt{\frac{1}{n^2x^2}+1}+\frac{1}{nx}\right)} = \frac{|x|}{\sqrt{\underbrace{\tfrac{1}{n^2x^2}}_{\to 0}+1}+\underbrace{\tfrac{1}{nx}}_{\to 0}} \to \frac{|x|}{\sqrt{1}} = |x| \text{ as } n \to \infty$$

Note limit is $|x|$ not x because $f_n(x) > 0$

(iii) $f_n : \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto \frac{1}{1+nx^2}$

Note $f_n(0) = 1, \forall n \quad \therefore f(0) = 1.$

For $x \neq 0$, $nx^2 \to \infty$ as $n \to \infty$.

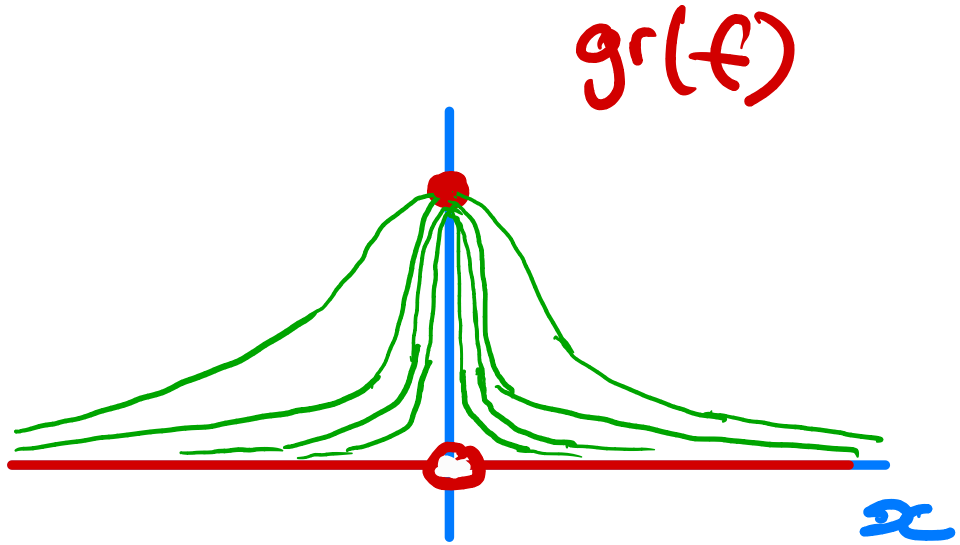
$$0 \leq \frac{1}{1+nx^2} \leq \frac{1}{nx^2}, \text{ and for fixed } x$$

$$\frac{1}{nx^2} = \frac{1}{n}\left(\frac{1}{x^2}\right) \to 0 \cdot \frac{1}{x^2} = 0$$

$\therefore$ By sandwich theorem $f_n(x) \to 0$ as $n \to \infty$,

i.e. $f(x) = 0$, for $x \neq 0$.

f is discontinuous at $x = 0$.



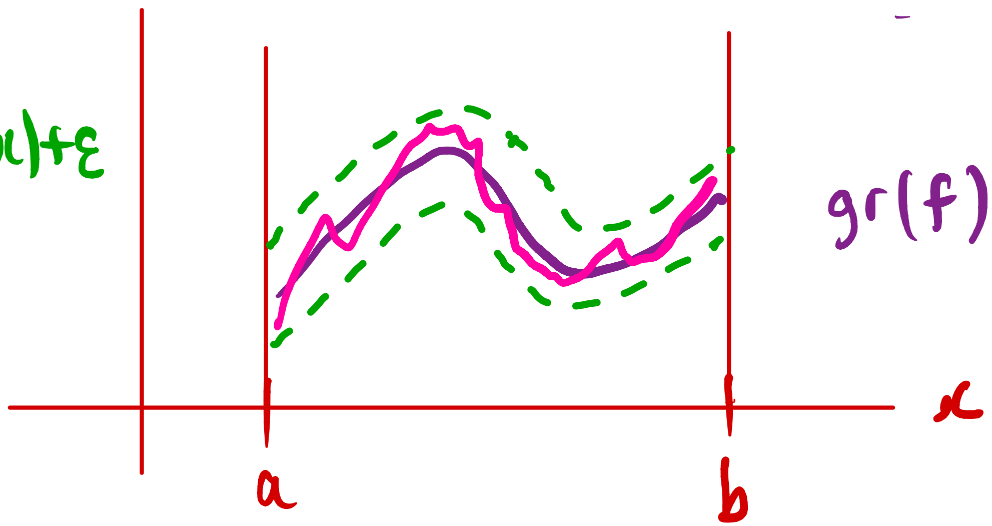
Def^n 9.1.3 (Uniform convergence)

We say the sequence $\{f_n\}$: $f_n : \Omega (\subseteq \mathbb{R}) \to \mathbb{R}$ **converges uniformly** to $f : \Omega \to \mathbb{R}$ on Ω if for every $\varepsilon > 0$, $\exists N$ such that

$$|f_n(x) - f(x)| < \varepsilon, \ \forall n > N \ \& \ \forall x \in \Omega.$$

Remark N depends only on ε & Ω.

$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$



$f_n \to f$

gr(f_n) lies inside the "ε-tube" for $n > N$

Ex 9.1.4 The functions $f_n(x) = x^n$ do not converge uniformly to $f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$

<u>Proof</u> Suppose it converges uniformly to $f(x)$.
Then $\forall \varepsilon > 0$, $\exists N$ such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \text{for all } n > N, \text{ and all } x \in [0,1]$$

Consider $\varepsilon = \frac{1}{4}$. Given uniform convergence, we have

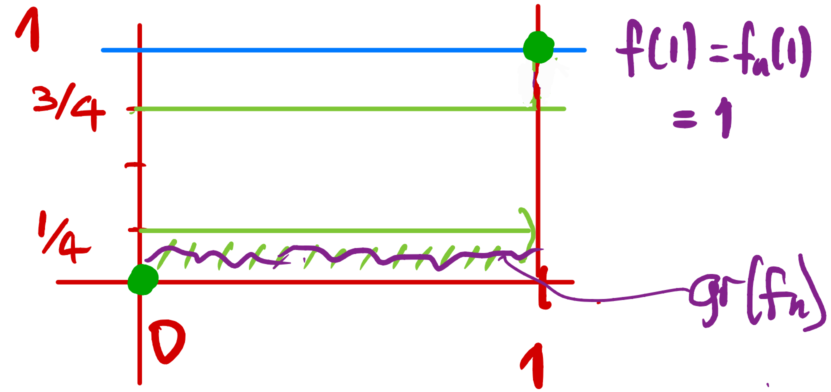
$$|f_n(x) - f(x)| < \frac{1}{4} \text{ for all } n > N \text{ and } x \in [0,1].$$

Split this condition into 2 parts

(i) $x = 1$, $f(1) = 1$ and $f_n(1) = 1$, $\forall n$

(ii) $x \in [0,1)$, $f(x)=0$

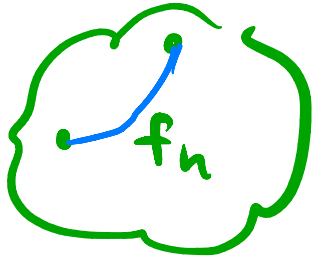
$\therefore |f_n(x) - f(x)| < \frac{1}{4} \Rightarrow 0 \leq f_n(x) < \frac{1}{4}, \ \forall n > N.$



However, f_n is continuous

and $f_n(0) = 0$, $f_n(1) = 1$ $\forall n$

$\therefore \exists c$ such that $f_n(c_n) = \frac{1}{2}$



Contradiction $f_n(x) < \frac{1}{4}$ for $x \in [0,1)$, $n > N$

$f_n(1) = 1$ for $x = 1$ $\forall n$.

$\therefore \nexists c_n$ such that $f_n(c_n) = \frac{1}{2}$ $\forall n > N$

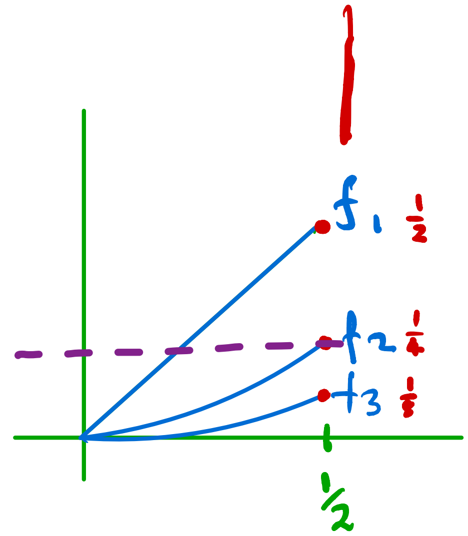
$\therefore$ f_n is not continuous $\times$

$\therefore$ the sequence f_n does **not** converge uniformly to f.

Important: If a sequence of continuous functions converges point-wise to the function f, and f is not continuous, then the sequence f_n does not converge uniformly to f.

Example 9.1.5

The function $f_n(x) = x^n$ converges uniformly for $f(x) = 0$ for $x \in [0, \frac{1}{2}]$.



Note $|f_n(x)| = |x^n| \leq \left(\frac{1}{2}\right)^n, \quad \forall x \in [0, \frac{1}{2}]$.

$\therefore |f_n(x)| \to 0$ as $n \to \infty$

$\therefore f(x) = 0, \quad x \in [0, \frac{1}{2}]$ pointwise limit function.

Given $\varepsilon > 0$, we have to find $N = N(\varepsilon)$ such that if $n > N$, then

$|f_n(x) - f(x)| = |f_n(x)| < \varepsilon$ for $\forall x \in [0, \frac{1}{2}]$.

So choose N such that $\left(\tfrac{1}{2}\right)^N < \varepsilon$.

i.e. $N\ln\left(\frac{1}{2}\right) < \ln(\varepsilon) \iff -N\ln(2) < \ln(\varepsilon)$

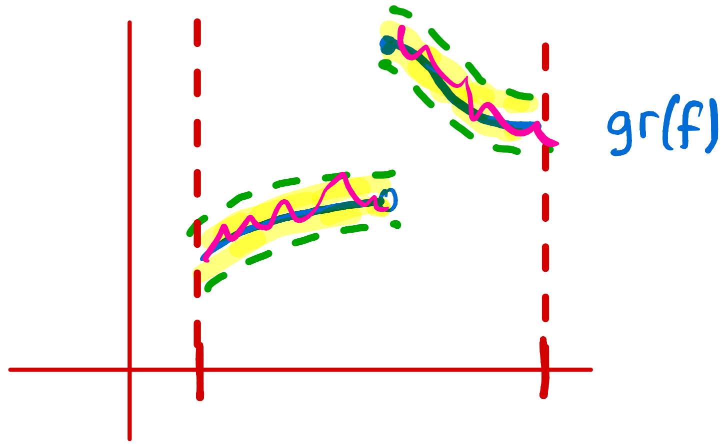
$$-\frac{\ln(\varepsilon)}{\ln(2)} < N$$

Similar argument for $x \in [0,a]$, $a<1$

$\therefore\ |f_n(x) - f(x)| < \varepsilon \ ,\ n > N > \dfrac{-\ln(\varepsilon)}{\ln(2)}$

$\therefore\ f_n(x) \to f(x)$ converges uniformly for $x \in [0, \tfrac{1}{2}]$.

if $f_n \xrightarrow{uc} f$ then f_n, $n>N$ has to lie in the yellow tube



Note, pictorially, if f_n converges UNIFORMLY to $f(x)$, then given $\varepsilon > 0$, $\exists N$ s.t. all f_n lie in an ε-nbd of f

Example 9.1.6

Prove this will work for any interval $[0,p]$, $\forall p<1$

Theorem 9.1.7

Consider a sequence of functions $f_n : [a,b] \to \mathbb{R}$ and suppose f_n's are continuous and $\{f_n\}$ converges uniformly to $f : [a,b] \to \mathbb{R}$, then f is also continuous.

Proof Given $\varepsilon > 0$, and $x, x_0 \in [a,b]$, we need to find a δ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Now

$$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$$

uc? $\quad ? < \varepsilon/3 \quad$ C $\quad ? < \varepsilon/3 \quad$ uc? $\quad ? < \varepsilon/3$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|.$$

(i) by uniform convergence, we can find an N such that $|f(x)-f_n(x)| < \varepsilon/3$, for $n > N$, $\forall x \in [a,b]$.

This implies (uniform convergence)

$$|f(x)-f_n(x)|, \text{ and } |f(x_0)-f_n(x_0)| < \frac{\varepsilon}{3} \quad , n > N.$$

(ii) f_n is a continuous function for $n \in \mathbb{N}$; so $\exists\, \delta$ such that

$$|x-x_0| < \delta, \quad |f_n(x)-f_n(x_0)| < \varepsilon/3 \,, \text{ for } x_0, x \in [a,b]$$

(iii) It now follows that if $|x-x_0| < \delta$:

$$|f(x)-f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \text{ and so } f \text{ is continuous.}$$

provided $|x-x_0| < \delta$ (ii)

The pointwise limit function f of a uniformly convergent sequence of continuous functions is continuous

Theorem 9.1.8 Let f_n be a seq^n of continuous fns on $[a,b]$ & suppose $f_n \to f$ uniformly, then

e.g. $f_n(x) = x^n$

$\int f_n(x) = \frac{x^{n+1}}{n+1}$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

" $\int_a^b \lim_{n\to\infty} f_n(x)\,dx$ "

Proof By theorem 9.1.7, f is a continuous function, and hence Rieman Integrable.

So the function $|f_n - f|$ is continuous, and therefore RI.

Given $\varepsilon>0$, $\exists N$: $|f_n - f| < \frac{\varepsilon}{(b-a)}$, $\forall x \in [a,b]$; and $n > N$.

It follows that

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \left| \int_a^b (f_n(x) - f(x))\,dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\, dx$$

$$< \int_a^b \frac{\varepsilon}{(b-a)}\, dx = \varepsilon$$

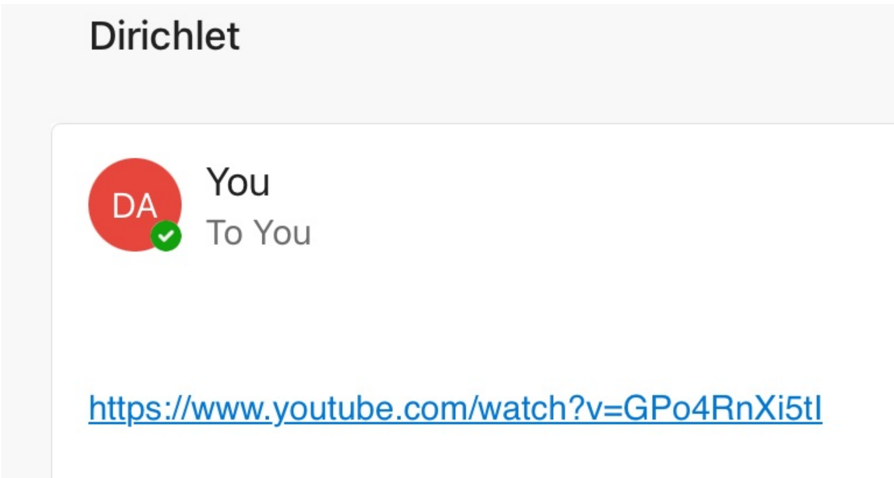
Hence $\forall \varepsilon > 0$, $\exists N$, such that $\forall n > N$

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| < \varepsilon$$

or $$\boxed{\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx} = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx$$

Obvious question: what are examples of sequences functions $f_n \to f$ with $\lim_{n\to\infty} \int f_n \neq \int f$?

The counterexamples require some explanation.


Dirichlet

You

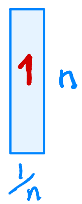
To You

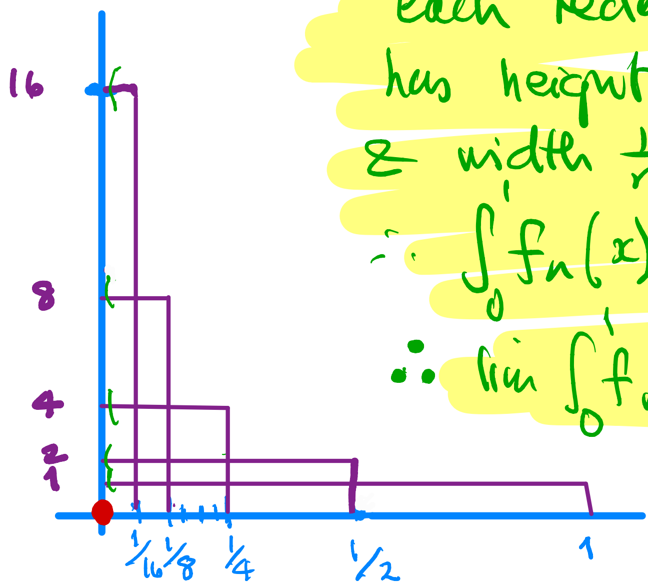
https://www.youtube.com/watch?v=GPo4RnXi5tI

This is a good explanation on youtube of unexpected behaviour for $\lim_{n\to\infty} \int f_n \neq \int f$ when $f_n \to f$!

Second example: $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} 0 & x = 0 \\ n & x \in (0, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1] \end{cases} \longrightarrow$$





each rectangle has height n & width $\frac{1}{n}$

$\therefore \int_0^1 f_n(x) \equiv 1$

$\therefore \lim \int_0^1 f_n(x) \equiv 1$

Can be shown using $L(f_n, P)$, $U(f_n, P)$ that

$$\int_0^1 f_n(x)\,dx \equiv 1 \quad \forall n. \quad \therefore \lim_{n\to\infty} \int_0^1 f_n(x) = 1.$$

Also what is the pointwise limit function $f(x)$ of $\{f_n(x)\}$?

Note that $f_n(0) = 0 \Rightarrow f(0) = \lim_{n\to\infty} f_n(0) = 0$.

If $x > 0$, then for $n > \frac{1}{x}$, i.e. $\frac{1}{n} < x$, $f_n(x) = 0$.

So $\lim_{n\to\infty} f_n(x) = 0 \quad \therefore f(x) = 0$

$$\therefore f(x) \equiv 0 \quad \text{and} \quad \int_0^1 f(x)\,dx = 0.$$

$$\therefore 1 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx = 0.$$

Note $f_n(x) \equiv 0$ for $n > \frac{1}{x}$ (directly from def^n of f_n)

$\therefore \{f_n(x)\}$ is <u>eventually</u> a sequence of '0's $\therefore f(x) = 0$

Completeness theorem for the real nos. $\mathbb{R}$.
(Real nos have no gaps or holes)

$\forall S \subseteq \mathbb{R}$, if S is bounded above, then $\sup S$ exists and $\sup S \in \mathbb{R}$.)

Related to this is the role of Cauchy sequences

A sequence $\{a_n\}_1^\infty$ is a Cauchy sequence iff given $\varepsilon > 0$, $\exists N$ such that

$$|a_n - a_m| < \varepsilon \quad , \quad \forall n, m > N.$$

Recall that a sequence $\{a_n\}_1^\infty$ converges if and only if $\{a_n\}_1^\infty$ is a Cauchy sequence

Theorem 9.1.9

A sequence of functions $\{f_n\}$, $f_n : \Omega \to \mathbb{R}$, converges uniformly to f if and only if for all $\varepsilon > 0$, $\exists N$ such that $\forall n, m > N$

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall n, m > N, \ \forall x \in \Omega$$

Proof ($\Rightarrow$) Suppose $f_n \to f$ uniformly, i.e.

$\forall \varepsilon > 0, \exists N$ such that for $\forall n > N$

$$|f_n(x) - f(x)| < \varepsilon/2, \quad \forall x \in \Omega.$$

$$\therefore \ |f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)|$$

$$\leq |f_m(x) - f(x)| + |f(x) - f_n(x)|$$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall m, n > N, \forall x \in \Omega$

$\therefore \{f_n\}$ is a Cauchy sequence.

($\Leftarrow$) Suppose $\{f_n(x)\}$ is a Cauchy sequence.

We know $\{f_n(x)\}$ converges pointwise for each $x \in \Omega$.

$\therefore$ let $\lim_{n\to\infty} f_n(x) = f(x)$. We claim $f_n \to f$ converges uniformly.

Given $\varepsilon > 0$, $\exists N$ such that $\forall m, n > N$

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2} \quad \forall x \in \Omega.$$

Let $n \to \infty$ & fix $m > N$ $\Rightarrow$ $|f(x) - f_m(x)| \leq \frac{\varepsilon}{2} < \varepsilon$.

i.e. $\{f_m(x)\}$ converges to $f(x)$.

Convergence to a limit & Cauchy condition are equivalent!

§9.2 Series of Functions

Compare with $S_k = \sum_{n=1}^{k} a_n$

$\lim_{k\to\infty} S_k = S = \sum_{n=1}^{\infty} a_n$

Def^n 9.2.1 (Series of functions)

Let $\{f_n\}$ be a sequence of functions, $f_n : \Omega \to \mathbb{R}$

$\forall k \in \mathbb{N}$, let $S_k(x) = \sum_{n=1}^{k} f_n(x) = f_1(x) + \cdots + f_k(x)$.

be the sequence of partial sums. We say

(i) $\sum_{n=1}^{\infty} f_n$ converges pointwise if $\{S_k\}$ converges pointwise

$$S_k(x) \to S(x) \quad (k \to \infty)$$

(ii) $\sum_{n=1}^{\infty} f_n$ converges uniformly if $\{S_k\}$ converges uniformly

$$S_k(x) \xrightarrow{u} S(x), \quad \forall x \in \Omega. \quad [k \to \infty \quad u = \text{uniform}]$$

Example 9.2.2.

$$\sum_{n=1}^{\infty} \frac{1}{(2+x^2)^n} \text{ converges uniformly. for } x \in \mathbb{R}$$

CORRECTION!

Proof We compute the partial sums

$$S_k(x) = \sum_{n=1}^{k} \frac{1}{(2+x^2)^n} = \frac{1}{(2+x^2)} \cdot \frac{1-\left(\frac{1}{2+x^2}\right)^k}{1-\left(\frac{1}{2+x^2}\right)}$$

geometric series

$a = \frac{1}{2+x^2}$, $r = \frac{1}{2+x^2}$

$S_n = a + ar + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$

$$= \frac{\frac{1}{(\cancel{2+x^2})}\left(1-\frac{1}{(2+x^2)^k}\right)}{(1+x^2)/(\cancel{2+x^2})} = \frac{1}{(1+x^2)}\left(1-\frac{1}{(2+x^2)^k}\right)$$

We note $\frac{1}{(2+x^2)^k} \to 0$ as $k \to \infty$ $\forall x \in \mathbb{R}$:

$\frac{1}{(2+x^2)^k} \le \frac{1}{2^k}$ and given $\varepsilon > 0$, $\frac{1}{2^k} < \varepsilon$

for $k > \ln(\varepsilon)/\ln(\frac{1}{2})$. Hence $\frac{1}{2^k} \to 0$ as $k \to \infty$.

$\therefore$ $S_k(x) \to S(x) = \frac{1}{1+x^2}$ as $k \to \infty$

For uniformity convergence: given $\varepsilon>0$,

We have $|s_k(x)-s(x)| = \dfrac{1}{(2+x^2)^{k+1}} \leq \dfrac{1}{2^{k+1}}$, $\forall x \in \mathbb{R}$.

and given $\varepsilon>0$, to ensure $\dfrac{1}{2^{k+1}} < \varepsilon$, we need

$$k > -\frac{\ln(\varepsilon)}{\ln(2)} - 1 \; (= K).$$

$\Rightarrow |s_k(x)-s(x)| < \varepsilon$ for $k>K$, and $\forall x \in \mathbb{R}$.

$$\therefore \sum_{n=1}^{k} \frac{1}{(2+x^2)^n} \to \frac{1}{1+x^2}^{*} \text{ uniformly as } k\to\infty.$$

equivalent statement: $s_k(x) \to s(x)^{*}$ uniformly as $k \longrightarrow \infty$.

DKA COMMENT This looked a surprising result to me! i.e. that somehow the series collapses to $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$!

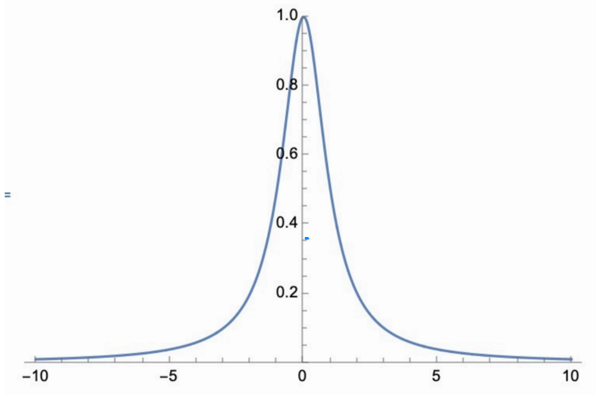
First simple check:

Let $x=0$: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2+0^2)^k} = \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1+0^2}$, ✓ so far, so good!

A more convincing check (using MATHEMATICA).

```
G[x_] := Sum[1 / (2 + x ^ 2) ^ k, {k, 1, 20}]
```

```
Plot[G[x], {x, -10, 10}, PlotRange → {0, 1}]
```

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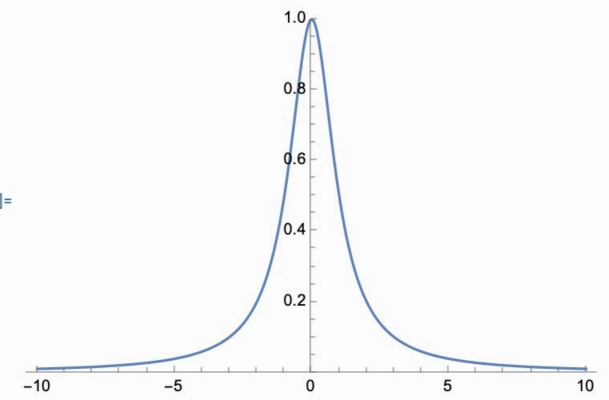
In[24]:= Plot[1 / (1 + x ^ 2), {x, -10, 10}, PlotRange → {0, 1}]
```

Out[24]=



So yes, I'm convinced! ☺

$$S_{20}(x) = \sum_{k=1}^{20} \frac{1}{(2+x^2)^k}$$

$$S(x) = \frac{1}{1+x^2}$$

not bad!

Theorem 9.2.3 Weierstrass M-test.

Suppose that there exists a sequence of non-negative numbers $\{M_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} M_n$ converges ($<\infty$) and also the sequence $\{f_n\}$, $f_n : \Omega \to \mathbb{R}$ satisfies

$$|f_n(x)| \leq M_n, \quad \forall x \in \Omega,$$

then

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly.}$$

Proof Let $S_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$

As $\sum_{n=1}^{\infty} M_n < \infty$, we have $\sum_1 M_n$ converges, we have the Cauchy condition:

$\sum_1 M_n$ converges.

$s_n \to s$, Cauchy $|s_n - s_m| < \varepsilon$

$\forall \varepsilon > 0, \exists N > 0$ such that $\quad \ell > N \quad n, m > N.$

$$|S_n - S_m| = S_n - S_m = \sum_{\ell=m+1}^{n} M_\ell < \varepsilon \qquad n > m \qquad M_\ell > 0.$$

$$\therefore \; |S_n(x) - S_m(x)| = \left|\sum_{\ell=m+1}^{n} f_\ell(x)\right|$$

$$\leq \sum_{\ell=m+1}^{n} |f_\ell(x)|$$

$$\leq \sum_{\ell=m+1}^{n} M_\ell < \varepsilon, \qquad \forall x \in \Omega$$

i.e. $|S_n(x) - S_m(x)| < \varepsilon, \quad \forall x \in \Omega, \quad n, m > N.$

$\therefore$ the partial sums are Cauchy and $\{S_n(x)\}$ converges uniformly by Thm 9.1.9.

Note: $0 \leq \dfrac{1}{(2+x^2)^k} \leq \dfrac{1}{2^k}$

Example 9.2.4

(i) Show that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly for all $x \in \mathbb{R}$.

$M_n = \frac{1}{n^2}$

Proof $\left|\frac{\sin(nx)}{n^2}\right| \le \frac{1}{n^2}, \ \forall x \in \mathbb{R}$

Let $M_n = \frac{1}{n^2}$ ($\sum \frac{1}{n^2}$ converges!) $\therefore$ ✓

(ii) Show $\sum_{n=1}^{\infty} \frac{1}{(2+x^2)^n}$ converges uniformly on $\mathbb{R}$

We note $\frac{1}{(2+x^2)^n} \le \frac{1}{2^n} \ (= M_n)$, $\forall x \in \mathbb{R}$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ converges

and so $\sum_{n=1}^{\infty} \frac{1}{(2+x^2)^n}$ converges uniformly.

Geom. Series

$a = \frac{1}{2} \quad r = \frac{1}{2}$

$S_k = \frac{1}{2} \frac{\left(1 - \left(\frac{1}{2}\right)^k\right)}{1 - \frac{1}{2}}$

$\to 1$ as $k \to \infty$