Thm 7.1.5 (Well-ordered)

WEEK 8

Let $g, f : [a,b] \longrightarrow \mathbb{R}$ be Riemann integrable. If $f(x) \leq g(x)$, $\forall x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.

Note $g-f = g + (-1)f$

Proof By Thm 7.1.4, $\int_a^b g-f = \int_a^b g + \int_a^b (-f)$ exists.*

Given $g(x) \geq f(x) \implies g(x) - f(x) \geq 0$, $\forall x \in [a,b]$, it follows, given partition $P = \{x_0, \ldots, x_n\}$ on $[a,b]$ of f, that $m_i = \inf_{[x_{i-1}, x_i]} g-f \geq 0$. (0 is a lower bound!)

i.e. $0 \leq L(g-f, P) \leq \int_a^b (g(x) - f(x))\,dx$ (which exists*)

$L(g-f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \geq 0$. $\forall P$

$\therefore \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Thm 7.1.6 Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable. then $|f|$ is also Riemann Integrable,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Existence is the issue
as $\int f(x)dx \le \int |f|(x)dx$
is implied by $f \le |f|$

Proof Let P be a partition of $[a,b]$, $(=\{x_0, x_1, \dots, x_n\})$

Recall the reverse triangle inequality

$$\big||a|-|b|\big| \le |a-b| \quad \text{and we have} \quad |f|(x) \overset{\text{DEF}}{=} |f(x)|$$

$a=5 \quad b=-7$

$$\text{So} \quad \big||f|(x)-|f|(y)\big| = \big||f(x)|-|f(y)|\big| \le |f(x)-f(y)|$$

For $x,y\in I_i=[x_{i-1},x_i]$:

$$M_i^{|f|}-m_i^{|f|} = \sup_{[x_{i-1},x_i]} \big||f|(x)-|f|(y)\big| \le \sup_{[x_{i-1},x_i]} |f(x)-f(y)| = M_i^f - m_i^f$$

i.e

$$\underset{x\in I_1}{\sup}\ |f(x)| - \underset{x\in I_1}{\inf}\ |f(x)| \le \underset{x\in I_1}{\sup} f(x) - \underset{x\in I_1}{\inf} f(x)$$

lub glb

$$\therefore\ U(|f|,P) - L(|f|,P) = \sum_{i=1}^{n}\left(M_i^{|f|} - m_i^{|f|}\right)(x_i - x_{i-1})$$

$$\le \sum_{i=1}^{n}\left(M_i^{f} - m_i^{f}\right)(x_i - x_{i-1}) = U(f,P) - L(f,P)$$

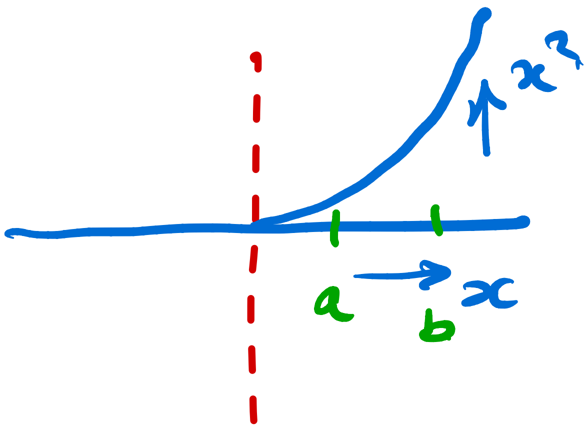
$$\Rightarrow\ U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P).$$

Since f is integrable, $\exists$ a partition P such that given $\varepsilon > 0$, $U(f,P) - L(f,P) < \varepsilon$, and hence $U(|f|,P) - L(|f|,P) < \varepsilon$, and $|f|$ is Riemann Integrable by R I C.

Note $f(x) \le |f|(x)\ \therefore \int f \le \int |f|$

Thm 7.1.7 Let $f:[a,b] \to \mathbb{R}$ be Riemann Integrable, then f^2 is Riemann Integrable. $\left(f^2(x) = f(x).f(x) \right)$

Proof As f is bounded $\Rightarrow \exists\ M$ such that $|f(x)| \leq M$. $\forall\ x \in [a,b]$. Let P be a partition of $[a,b]$, then



Note

$$M_i f^2 = M_i(|f|^2) = (M_i |f|)^2$$

$$m_i f^2 = m_i |f|^2 = (m_i |f|)^2$$

$$\therefore\ M_i f^2 - m_i f^2 = (M_i |f|)^2 - (m_i |f|)^2$$

$$= \left[M_i(|f|) - m_i(|f|)\right]\left[M_i(|f|) + m_i(|f|)\right] \geq 0$$

M = bound for f on [a,b].
$|f(x)| \leq M$

$$\leq 2M\left(M_i(|f|) - m_i(|f|)\right)$$

$|f|(x) \geq 0$
0 l.b.

$$\therefore U(f^2,P) - L(f^2,P) \leq 2M \sum_{i=1}^{n} \left(M_i(|f|) - m_i(|f|)\right)(x_i - x_{i-1})$$

$$\leq 2M\left[U(|f|,P) - L(|f|,P)\right]$$

Now f is RI $\Rightarrow$ $|f|$ is RI by Thm 7.1.6

$\therefore$ Given ε, $\exists$ partition P such that

$$U(|f| P) - L(|f|,P) < \varepsilon/2M$$

i.e. $U(f^2,P) - L(f^2,P) < \varepsilon$ and f^2 is RI

Theorem 7.1.8 If $f, g : [a,b] \to \mathbb{R}$ are RI, then $f \cdot g$ is RI.

Proof Note $f \cdot g = \dfrac{(f+g)^2 - (f-g)^2}{4}$

Use "$f+g$, f^2, cf" Theorems on RI

f^2 RI
$(f+g)^2$ RI
$(-1) \cdot g$ RI
$f + (-1) g$ RI

Theorem 7.1.9 (Mean value theorem for integrals)

Let $f : [a,b] \to \mathbb{R}$ be continuous, then $\exists\ c \in (a,b)$ such that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Proof If f is constant, the statement is obvious

If $f(x) \equiv k$, a constant, then $\int_a^b f(x)\,dx = k(b-a)$, so any choice of $c \in (a,b)$, will do.
For any partition P, with $f(x) \equiv k$: $L(f,P) = k(b-a) = U(f,P)$

Let us assume f is <u>not</u> constant, then

$m < M$, where $m = \inf_{x \in [a,b]} f(x)$ and $M = \sup_{x \in [a,b]} f(x)$

By continuity of f and boundedness principle

$\exists\ x_m, x_M$ such that $f(x_m) = m$ & $f(x_M) = M$.

Let us assume $x_m < x_M$, (and not equal!)

$\therefore\ m = f(x_m) \leq f(x) \leq f(x_M) = M$

By Thm 7.1.5

$$\int_a^b m \leq \int_a^b f \leq \int_a^b M, \qquad m, M \text{ constants.}$$

Integrating gives

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a),$$

dividing by $(b-a)$

$$f(x_m) = m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M = f(x_M)$$

By the Intermediate value theorem, $\exists$ exists $c \in (a,b)$ such that

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a},$$

$c \in (a,b)$ actually lies on (x_m, x_M) or (x_M, x_m)

i.e. c lies between x_M and x_m !

§8 Fundamental theorem of Calculus.

WEEK 8 cont.

§8.1 FTC

Def^n 8.1.1 (Anti-derivative).

A function F is an anti-derivative of f if

$$\frac{dF}{dx} = f(x)$$

Thm. 8.1.2 (FTC)

Assume $f : [a,b] \to \mathbb{R}$ is continuous. Define

$$F(x) = \int_a^x f(t)\,dt$$

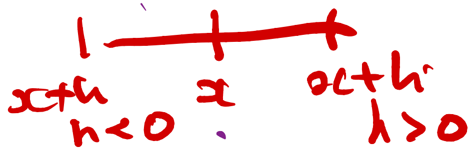
then F is an anti-derivative of f and

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

<u>Proof</u> We claim $F'=f$ on (a,b). Let us take $x\in(a,b)$ & choose h small so $x+h\in(a,b)$.

Consider
$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{\int_a^{x+h}f(x)\,dx-\int_a^{x}f(x)\,dx}{h}$$

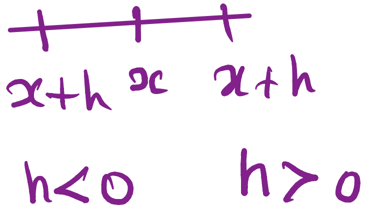
$$=\lim_{h\to 0}\frac{\int_x^{x+h}f(t)\,dt}{h}=\lim_{h\to 0}f(c_h),$$ where $c_h\in(x,x+h)$.



As $h\to 0$, $c_h\to x$, and continuity of f gives

$$F'(x)=\lim_{h\to 0}f(c_h)=f(x).$$

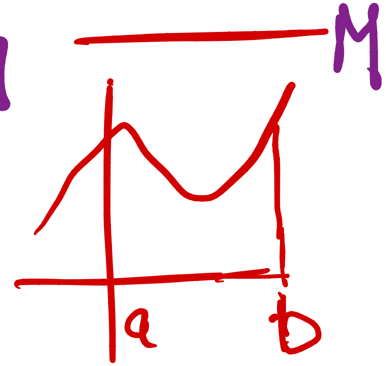
cont of f



Theorem 8.1.3 Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable & define $F:[a,b]\to\mathbb{R}$ by

$$F(x)=\int_a^x f(t)\,dt,$$ then F is continuous

Proof Since f Riemann integrable, f is bounded; then $\exists\ M$ s.t. $|f(x)|\le M,\ \forall x\in[a,b]$



Let $x, x_0\in[a,b]$.

$$|F(x)-F(x_0)| = \left|\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt\right|$$

$$= \left|\int_{x_0}^{x} f(t)\,dt\right| \le M\,|x-x_0| < M\cdot\frac{\varepsilon}{M}$$

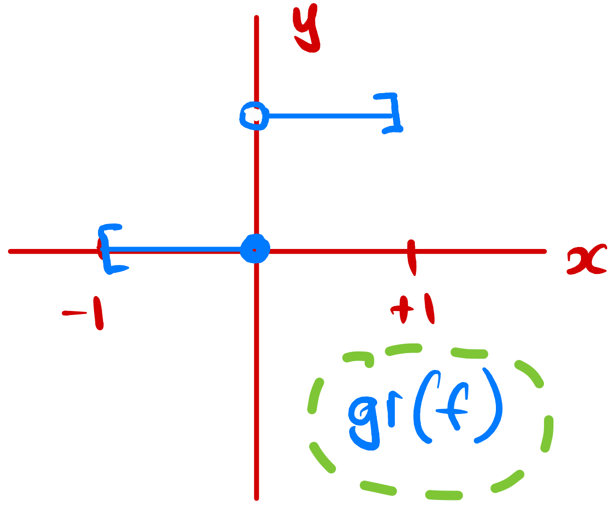
$\therefore$ If $|x-x_0|<\delta=\varepsilon/M$, then $|F(x)-F(x_0)|<M\cdot\frac{\varepsilon}{M}=\varepsilon$

and so F is continuous at $x=x_0$.

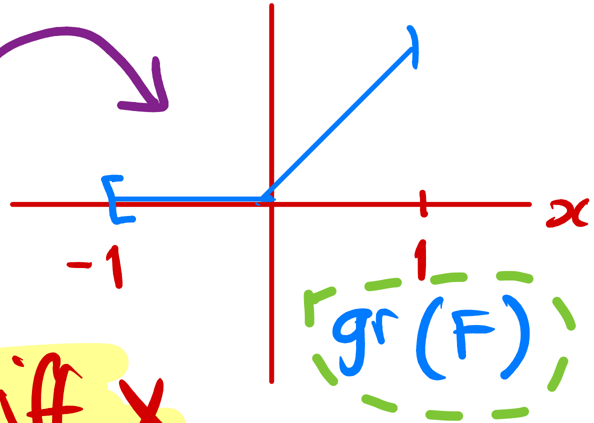
Example 8.1.4

Let $f:[-1,1]\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}0 & ; x\in[-1,0]\\ 1 & ; x\in(0,1]\end{cases}$$



Note $F(x)=\int_{-1}^{x} f(t)\,dt=\begin{cases}0 & x\in[-1,0]\\ x & x\in(0,1]\end{cases}$



The function F is continuous but not differentiable at $x=0$

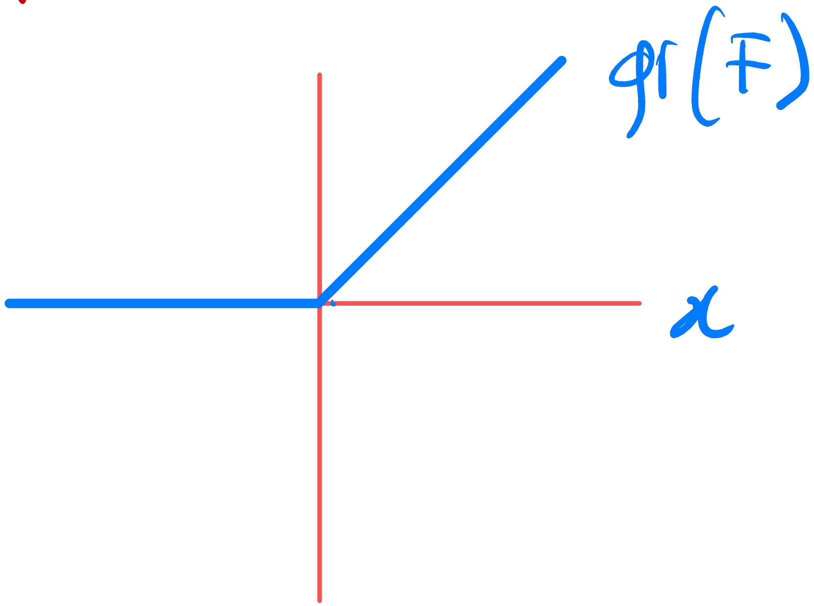
$$\left\{\lim_{x\to0^+}F'(x)=1\right\}\neq\left\{\lim_{x\to0^-}F'(x)=0\right\}$$ Diff ✗

$$\lim_{x\to0^+}F(x)=\lim_{x\to0^-}F(x)=F(0)=0.$$ Cont ✓

$$F(x)=\int_{-1}^{x} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx \qquad x>0$$

$$= 0 + x.$$

$$F(x) = \int_{-1}^{x} f(x)\,dx = 0 \qquad x \leq 0$$



Corollary 8.1.5

Every continuous function f has an anti-derivative.

Proof The function f is Riemann Integrable.
and by FTC, $F(x) = \int_0^x f(t)\,dt$ is an anti-derivative of f.

Def^n 8.1.6 Indefinite integral

If F is an anti-derivative of f, we define
$$\int f(x)\,dx = F(x) + c \quad , c \text{ constant}$$
as the indefinite integral of f.

Note

(i) If both F & G are antiderivatives of f
$F(x) = G(x) + C$, for some constant C.

$$(F(x)-G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$$

$\therefore\ (F(x)-G(x))' = 0 \Rightarrow F(x) - G(x) = C$, constant.

(ii) $\int_a^b f(t)\,dt = G(b) - G(a) = F(b) - F(a)$.

<u>Theorem 8.1.7</u> If f, g have anti-derivatives on $[a,b]$, then so do $f+g$, cf, $\forall c \in \mathbb{R}$.

<u>Proof</u> $(F+G)' = F' + G' = f + g$

$F+G$ is anti-derivative of $f+g$

& $(cF)' = cF' = cf$.

cF is antiderivative of cf.

Theorem 8.1.8 (Integration by Parts)

Let $f, g : [a,b] \to \mathbb{R}$ be differentiable. If fg' has an antiderivative, then $f'g$ has an anti-derivative and

$f(x) \cdot g(x)$

$$\int f'(x) g(x)\,dx = -\int g'(x) f(x)\,dx + f(x) \cdot g(x)$$

Proof Let H be the antiderivative of $h = fg'$

i.e. $H' = h = fg'$

Now $(fg)' = f'g + fg'$ – differentiation.

$$\therefore f'g = (fg)' - fg' = (fg)' - H' = (fg - H)'$$

$\therefore$ Anti derivative of $f'g$ is $fg - H$

END OF WK 8