Differential and Integral Analysis

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Weeks 1–4

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Room: MB-127

Teaching: Analysis, Algebra, Control Theory
Coding, Dynamical Systems,
Differential Topology (Pure & Applied!)

Research: Dynamical Systems and Networks
PI for both UK and EU research grants.

Chapter 1 Revision

1.1 Continuity

Let $\Omega \subseteq \mathbb{R}$ be a domain (an interval or all of $\mathbb{R}$).

$f : \Omega \to \mathbb{R}$

Defn 1.1.1 Pointwise continuity

Let $f : \Omega \to \mathbb{R}$, then the function f is continuous at $a \in \Omega$ if for any $\varepsilon > 0$, $\exists$ $\delta > 0$ (δ usually depends on (a, ε)) such that $|f(x) - f(a)| < \varepsilon$ for $|x - a| < \delta$.

Remark **Note** δ typically depends on both ε and the pt $a \in \Omega$.

Some examples of using the definition can be simple. e.g.

<u>Example</u> Prove $f(x) = x$ $(f: \mathbb{R} \to \mathbb{R})$ is a continuous function.

<u>Proof</u> Consider continuity at $a \in \mathbb{R}$. Then

$$|f(x) - f(a)| = |x-a| \text{ and given } \varepsilon > 0,$$

we need to find a δ such that

$$|f(x) - f(a)| < \varepsilon \text{ when } |x-a| < \delta$$

But $|f(x) - f(a)|$ <u>is</u> $|x-a|$!!!

So $\underbrace{|f(x) - f(a)| < \varepsilon}_{|x-a|<\varepsilon} \iff \underbrace{|x-a| < \varepsilon}_{|x-a|<\delta=\varepsilon}$

Choose $\delta = \varepsilon$

Example Prove $f(x) = x^2$, $x \in \mathbb{R}$, is continuous at $x = 0$. $a = 0$

Proof Consider $|f(x) - f(0)| = |x^2| = x^2$. Given $\varepsilon > 0$,

we need to find $\delta > 0$ such that

$$|x - 0| < \delta \text{ implies } |x^2 - 0| < \varepsilon.$$

$$|x^2| < \varepsilon, \Rightarrow |x|^2 < \varepsilon, \text{ i.e. } |x| < \sqrt{\varepsilon}.$$

Choose $\delta = \sqrt{\varepsilon}$.

Simpler qu?

Ex $f(x) \equiv 0$ Continuity at $x = a$.

$|f(x) - f(a)| = 0 < \varepsilon$, Any $\delta > 0$ will do.

Example Continuity of $f(x) = |x|$, $x \in \mathbb{R}$.

Note $|f(x) - f(a)| < ||x| - |a||$

✓(i) $a = 0$: $|f(x) - f(a)| = |x|$ & $|x - a| = |x|$

?(ii) $a > 0$, for $x > 0$ $|f(x) - f(a)| = |x - a|$ Choose $\varepsilon <$??

?(iii) $a < 0$, for $x < 0$ $|f(x) - f(a)| = |x - a|$ Choose $\varepsilon <$??

(i) Easy!

(ii) & (iii) Step 1 choose $\delta = \frac{|a|}{2}$

Step 2 obtain $|f(x) - f(a)| < k|x - a|$, $k = ?$

Choose $\delta = \min\left\{\frac{|a|}{2}, \frac{\varepsilon}{k}\right\}$.

Example Consider $f(x) = x^2$. Show $f(x)$ is continuous at $a > 0$.

Given $\varepsilon > 0$, we want to show that

find δ ?

how do I bound this ?

if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

First of all, $|f(x) - f(a)| = |x^2 - a^2| = |x+a|\,|x-a|$

This is good because we want to make $|x-a| < \delta$ and it appears explicitly!

But we have to deal with $|x+a|$!

δ's at most 1

Step 1 Suppose we decide to restrict $|x-a|$ to be less than 1

then if $x \in (a-1, a+1)$. $\qquad a-1 < x < a+1$

$\Rightarrow \quad x \in (-a-1, a+1) \quad \Rightarrow |x| < a+1$

$\Rightarrow \quad |x+a| < |x| + |a| < a+1+|a| = 2a+1 \quad (a>0)$

So with this "$\delta = 1$" restriction we have

$$|x^2-a^2| < |x+a|\,|x-a| < (2a+1)|x-a|$$

i.e. $|x^2-a^2| < (2a+1)\,|x-a|$

Step 2 But, we can now make $|x^2-a^2|<\varepsilon$, provided

$$(2a+1)|x-a|<\varepsilon,$$

$\therefore$ provided $|x-a|<\dfrac{\varepsilon}{2a+1}$ and $|x-a|<1$

$$\boxed{\text{Choose } \delta=\min\left\{\frac{\varepsilon}{2a+1},1\right\}}$$

Ex $f(x)=x^3$ continuous at $x=a$.

Easy: $a=0$

More difficult $a\neq 0$:

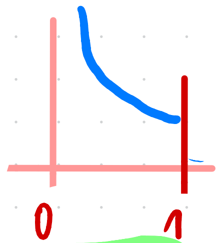
$$f(x)-f(a)=(x-a)(x^2+ax+a^2)$$

How do you bound x^2+ax+a^2? cf. $x+a$ above.

Example Consider $f(x) = 1/x^2$ defined on $\mathbb{R} \setminus \{0\}$

Show $f(x)$ is continuous at any $a \in \mathbb{R} \setminus \{0\}$.

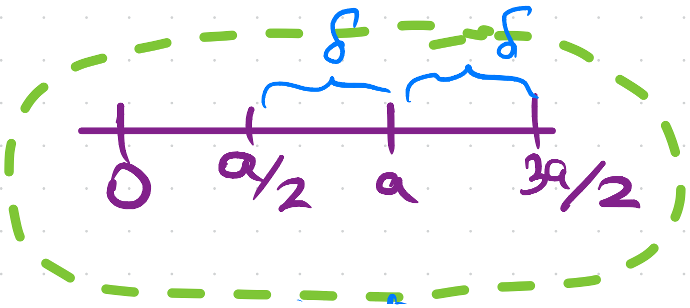
Consider $|f(x) - f(a)| = \left| \frac{1}{x^2} - \frac{1}{a^2} \right| = \left| \frac{a^2 - x^2}{x^2 a^2} \right|$



$$= \frac{|x-a||x+a|}{x^2 a^2} = |x-a| \frac{|x+a|}{x^2 a^2}$$

we need to bound this

STEP 1 Let $a > 0$. Suppose we consider $x \in \left(\frac{a}{2}, \frac{3a}{2}\right)$



i.e. $|x-a| < \frac{a}{2}$ i.e. "$\delta = a/2$".

So $\frac{|a|}{2} < x < \frac{3|a|}{2}$

$\Downarrow$ $\qquad$ $\Downarrow$

$\frac{1}{|x|} < \frac{2}{|a|}$ $\qquad$ $|x| < \frac{3|a|}{2}$

$\Downarrow$

$$|x+a| \le |x| + |a| < \frac{5|a|}{2}$$

STEP 2

$$|f(x)-f(a)| < |x-a| \cdot \frac{\left(\frac{2}{|a|}\right)^2 \frac{5|a|}{2}}{|a|^2}$$

$\therefore$

$$= |x-a| \frac{10}{|a|^3} < \delta \frac{10}{|a|^3}$$

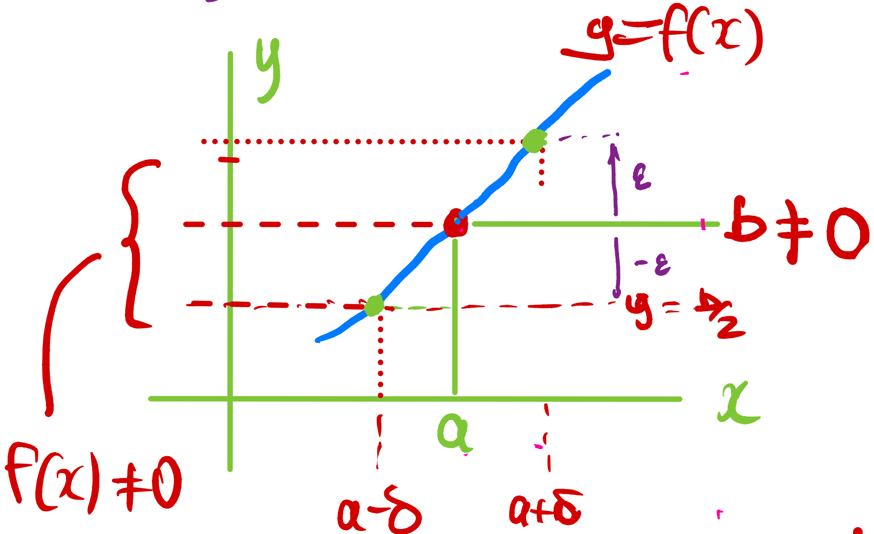
which is less than ε if $\delta < \frac{\varepsilon |a|^3}{10}$

so choose $\delta = \min\left\{\frac{|a|}{2}, \frac{\varepsilon |a|^3}{10}\right\}$

to accommodate step1 and step 2.

Theorem 1.1.2

If $f : \Omega \to \mathbb{R}$ is continuous at $a \in \Omega$ & $f(a) \neq 0$, then $f(x) \neq 0$ in an open nbd of a, i.e. an interval (c,d) s.t. $a \in (c,d)$.



Let $b = f(a)$. Choose $\varepsilon = \frac{|b|}{2}$

$$\Rightarrow |f(x) - f(a)| < \varepsilon \Rightarrow |f(x) - f(a)| < \frac{|b|}{2}$$

$$b - \frac{|b|}{2} < f(x) < b + \frac{|b|}{2}$$

NOTE: $0 \notin I = \left(b - \frac{|b|}{2}, b + \frac{|b|}{2}\right)$ for $b \neq 0$.

We have f is continuous at $x = a$. Therefore there is a $\delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon \left(= \frac{|b|}{2}\right)$ for $|x - a| < \delta$ and $f(x) \neq 0$ in the interval I.

Thm 1.1.3 (Boundedness Principle)

Let $f : [a,b] \to \mathbb{R}$, be a real-valued continuous function on the closed, bounded interval $[a,b]$, then f attains its maximum and minimum

$a \leq x \leq b$

i.e. $\exists\, c \in [a,b]$ s.t. $f(c) = \min\limits_{x\in[a,b]} f(x) = m$ (say)

and $\exists\, d \in [a,b]$ s.t. $f(d) = \max\limits_{x\in[a,b]} f(x) = M$ (say)

Furthermore, if $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$

$\exists$ sequences $\{x_n\}, \{y_n\} \subseteq [a,b]$ s.t.

$$\lim_{n\to\infty} x_n = m \quad \& \quad \lim_{n\to\infty} y_n = M.$$

Infimum & Supremum

Let $S \subseteq \mathbb{R}$ suppose $\exists\ s_1 \leq s\ ,\forall\ s \in S$

Then s_1 is a. LOWER BOUND lb

inf S = the greatest LOWER BOUND (glb).

property of inf S ($= l$). ($=$ glb(S))

(i) $l \leq s\ ,\forall\ s \in S$

(ii) if $l' > l$, then l' is NOT a lower bound.

Suppose $\exists\ s_2 \geq s.\ \forall s \in S$

Then s_2 is an UPPER BOUND ub

sup S = the least UPPER BOUND (ub)

property of sup S ($= u$) called the (lub)

(i) $s \leq u, \forall s \in S$

(ii) if $u' < u$, then u' is NOT an upper bound.

Ex $f(x)=\frac{1}{x}$ on $\Omega=(0,1)$. Find inf and sup of the set $S=\{\frac{1}{x}, x\in(0,1).\}$

So $x\in S \Rightarrow 1 > x > 0 \Rightarrow \frac{1}{x} > 1$, i.e. $f(x) > 1$

$\therefore$ 1 is a LB of S

In fact 1 is GLB of S

$\because$ if $l' > 1$, then l' is NOT a LB

Note any $x\in S$ s.t. $1 > x > \frac{1}{l'} \Rightarrow l' > \frac{1}{x} > 1$ and $\frac{1}{x}\in S$

$\therefore$ l' not an upper bound $\Rightarrow 1 = LUB(S)$

Also note given $n\in\mathbb{N}$, if we choose $x_n=\frac{1}{n}<1$ in S

then $\frac{1}{x_n}=n \in S$, for any $n\in\mathbb{N}$.

$\therefore$ $\exists$ exist arbitrarily large real nos in S and $\therefore$ No upper bound

A NOTE ON THE Completeness of the reals

All sets $S \subseteq \mathbb{R}$ bounded below have an inf (greatest lower bound)

above sup (least upper bound)

The existence of inf for sets bounded below and sup for sets bounded above is called completeness.

The real numbers $\mathbb{R}$ "have no holes or missing numbers!!"

NOT A DEFN – Zero marks in an exam – but a useful phrase!!

Example $S_1 = \{x \mid 0 < x < 1\}$ – $\inf(S_1)$ and $\sup(S_1)$?

$S_2 = \{\frac{1}{n}, n \in \mathbb{N}\}$ – $\inf(S_2)$ and $\sup(S_2)$?

Are the infs and sups minima or maxima of S_1 or S_2? Remember $\ell = \inf(S)$ is a minimum if $\ell \in S$

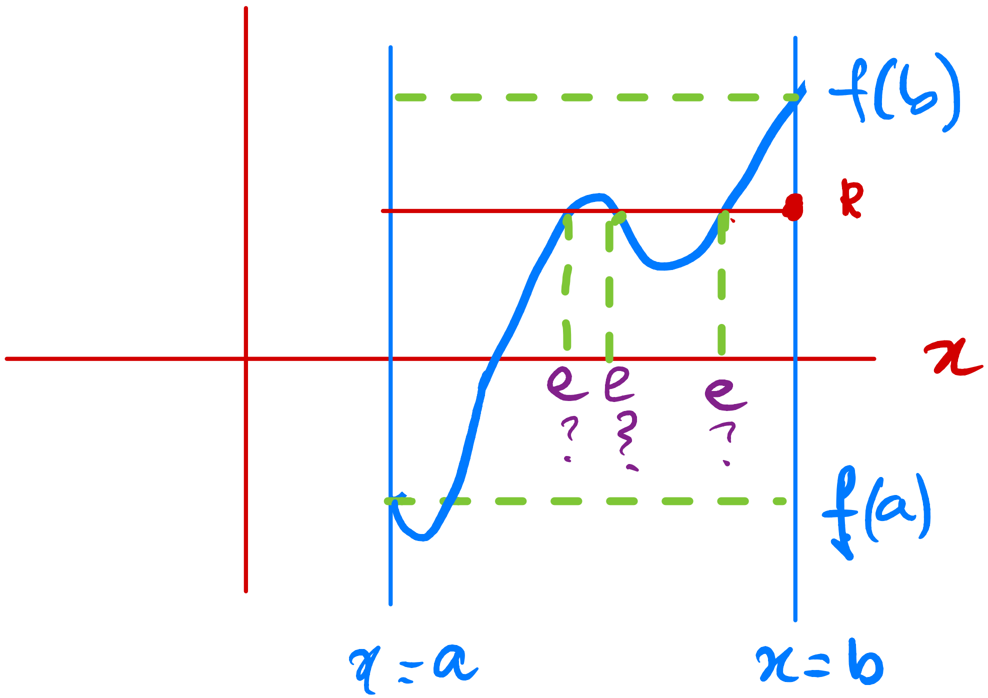
Similarly for a maximum.

Thm 1.1.4 (Intermediate Value theorem)

Let $f : [a,b] \to \mathbb{R}$ be a real valued continuous function on the closed bounded interval $[a,b]$. then f assumes all of its values between $f(a)$ and $f(b)$

i.e. if $k \in (f(a), f(b))$ or $(f(b), f(a))$ $\exists$ $c \in [a,b]$ s.t. $f(c) = k$

The graph of the function has no "breaks"!



§2 Differentiation

$(a,b) = \{x \mid a < x < b\}$

$(a,b) \subseteq \mathbb{R}$.

Def^n 2.1.1 The derivative

Let $x_0 \in (a,b)$, $f:(a,b) \longrightarrow \mathbb{R}$ a real valued function. The derivative of f at $x = x_0$ is defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad ✓$$

$$\left(= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) ✓$$

$x - x_0 = h$

$x = x_0 + h$

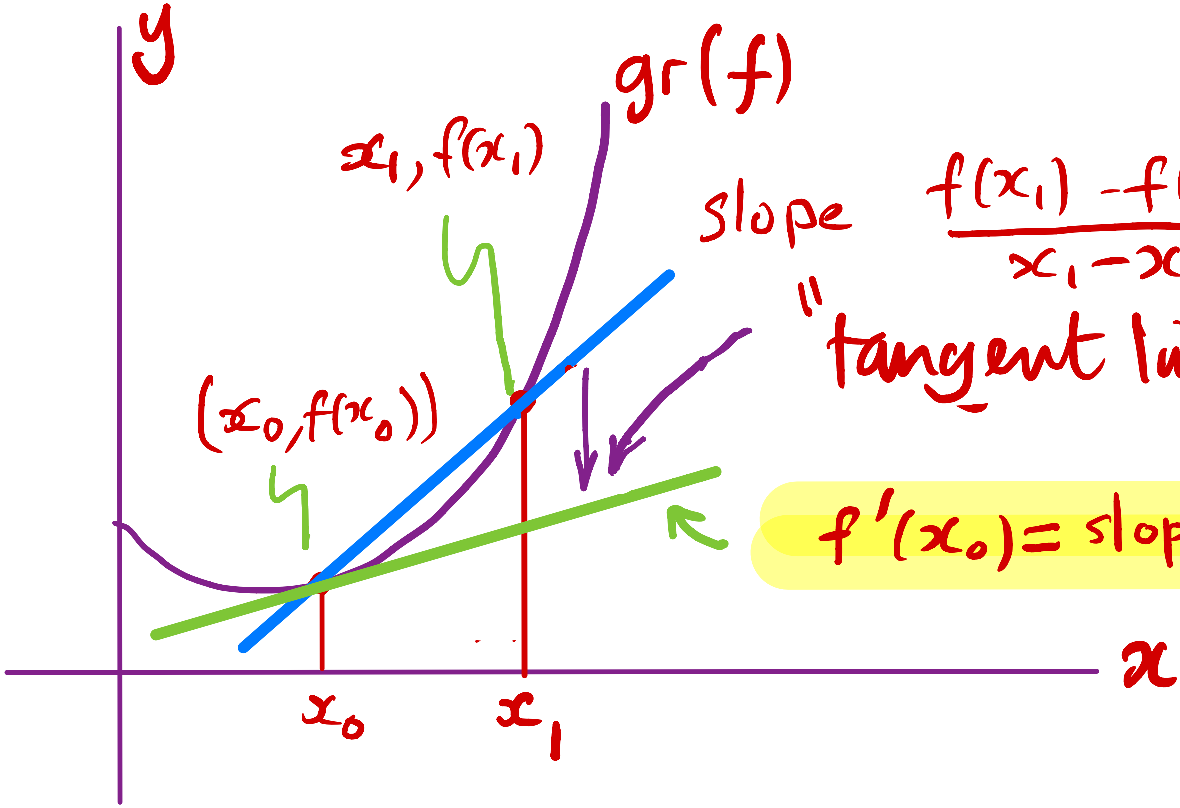
The function is said to be <u>differentiable</u> if $f'(x)$ exists for every $x \in (a,b)$.

$y = f(x)$: $\dfrac{\Delta y}{\Delta x}$

Remark Geometrically, the derivative $f'(x_0)$ is the slope of the tangent to the graph of $y = f(x)$ in the (x,y) plane.

$f(x) = |x|$

$y = |x|$



slope $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

"tangent line"

slopes

$f'(0) \nexists$

$f'(x_0) =$ slope of tangent at $x = x_0$

<u>Proposition 2.1.2</u> If f is differentiable at x_0, then f is continuous at $x = x_0$.

<u>Proof</u> $\boxed{\lim_{x \to x_0} f(x)} = \lim_{x \to x_0} \left(\underbrace{f(x_0)}_{\to f(x_0)} + \left(\underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\to f'(x_0)} \right) \times \underbrace{(x - x_0)}_{\to 0} \right)$

$\| $

$f(x_0)$??

"for continuity"

$$= f(x_0) + \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) (x - x_0)$$

$$= f(x_0) + f'(x_0) \cdot 0 \quad \boxed{= f(x_0)}$$

Therefore f is continuous at $x = x_0$. ✓

Example Compute $f'(x)$ for $f(x) = x^2$ at $x = a$

Proof
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-a^2}{x-a} = \lim_{x\to a} \frac{(x+a)(x-a)}{(x-a)}$$
$$= \lim_{x\to a} (x+a) = 2a.$$

$$\left(\lim_{x\to a} f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\therefore \boxed{f'(x)\Big|_{x=a} = 2a}$$

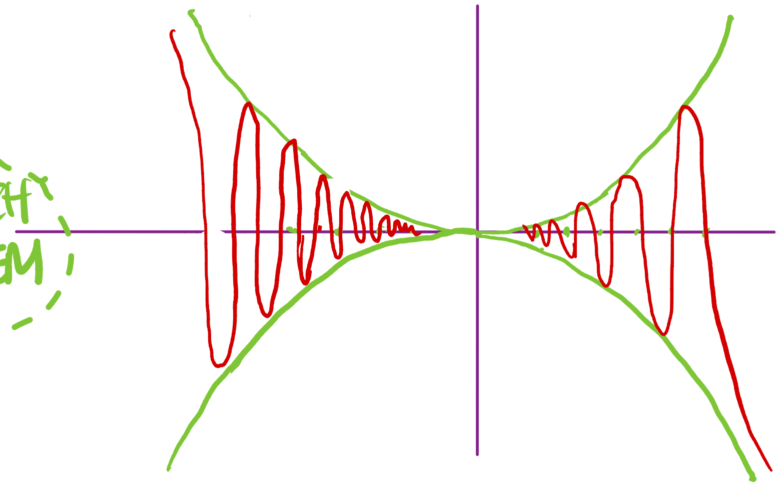
Example Show that $f'(0)=0$ for the function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$

Consider $\dfrac{f(x)-f(0)}{x-0} = \dfrac{x^2 \sin\left(\frac{1}{x}\right)}{x}$

$|x||\sin\frac{1}{x}|$ $(x \neq 0)$

$$= x\sin\left(\tfrac{1}{x}\right)$$

$$|f'(0)| = \lim_{x\to 0} \left|\frac{f(x)-f(0)}{x-0}\right|$$
$$= \lim_{x\to 0} \left|x \cdot \sin\left(\tfrac{1}{x}\right)\right|$$



SANDWICH THEOREM

But $\lim_{x\to 0} 0 \leq \lim_{x\to 0} \left|x . \sin\left(\tfrac{1}{x}\right)\right| \leq \lim_{x\to 0} |x|$

$\underset{0}{\parallel}$ ✓ $\therefore \Rightarrow \underset{0}{\parallel}$ $\quad$ $\underset{0}{\parallel}$ ✓ $\Rightarrow |f'(0)| = 0 \Rightarrow \boxed{f'(0) = 0}$ ✓

$$\left|\frac{f(x)}{x}\right| = x^{\alpha-1}$$
$\alpha > 1$

$f(x) = x^{\alpha}\sin\left(\tfrac{1}{x}\right)$

Def^n Suppose f and g are two functions such that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$,

then f is said to converge faster than g if

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = 0.}$$

Example Let $f(x) = (x-a)^3$, $g(x) = x - a$

$f(x)$ converges to zero faster than $g(x)$ since

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{(x-a)^3}{(x-a)} = \lim_{x \to a} (x-a)^2 = 0.$$

$$\boxed{f(x) = o(g(x)) \text{ meaning } \lim_{x \to a} \frac{f(x)}{g(x)} = 0}$$

(LANDAU NOTATION)

Lemma 2.1.4

The function $f : (a,b) \to \mathbb{R}$ is differentiable **iff** $\ast$

$\exists$ $B, m \in \mathbb{R}$ and a function $r_f : (a,b) \to \mathbb{R}$.

such that, (i) $f(x) = m(x - x_0) + B + r_f(x)$ $\ast$

(ii) $r_f(x) = o(x - x_0)$, i.e. $\lim\limits_{x \to x_0} \dfrac{r_f(x)}{x - x_0} = 0$.

Proof

$\Rightarrow$ Assume f is differentiable and $B = f(x_0)$ and $m = f'(x_0)$

Define $r_f(x) = f(x) - f'(x_0)(x - x_0) - f(x_0)$,

then

$$\lim_{x \to x_0} \frac{r_f(x)}{x - x_0} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - \frac{f'(x_0)(x - x_0)}{(x - x_0)} \right) = 0$$

Conversely, suppose (1) holds and $r = o(x - x_0)$

Substituting $x = x_0$ in (1) we get $B = f(x_0)$

$$\lim_{x \to x_0} \frac{r_f(x)}{(x - x_0)} = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - m \right) = 0$$

It follows that if $\lim_{x \to x_0} \frac{r_f(x)}{(x - x_0)} \to 0$,

then $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)} = m \Rightarrow m = f'(x_0)$

END OF WEEK 1