MATH 5105 Differential and Integral Analysis
Exercise Sheet 6

Coursework Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Show that f is uniformly continuous.

   Proof. Consider $x, y \in \mathbb{R}$. Then we know by Lemma 3.1.5 of the Lecture notes that if $|f'(\xi)| \leq M$ for all $\xi \in \mathbb{R}$. Then $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Therefore if we choose $\delta(\varepsilon) = \frac{\varepsilon}{M}$ then

   $$|f(x) - f(y)| \leq M|x - y| < \varepsilon.$$

   $\square$

2. Consider $f(x) = \frac{1}{x^2}$ on $[a, \infty)$ for $a > 0$. Show that f is uniformly continuous.

   Proof. We compute $|f'(x)| = \left|-\frac{2}{x^3}\right| \leq \frac{2}{a^3}$. Hence f is differentiable with bounded derivative and hence uniformly continuous. $\square$

Problems

3. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x, g : \mathbb{R} \to \mathbb{R}, g(x) = \sin(x)$. Prove or disprove the following statements

   (a) f is uniformly continuous,

   Proof. TRUE: f is uniformly continuous. As $f'(x) = 1$ for all $x \in \mathbb{R}$, we have $|f'(x)| \leq M$ with $M = 1$ so f has bounded derivative and hence uniform continuity follows from 1. $\square$

   (b) g is uniformly continuous.

Proof. TRUE: g is uniformly continuous
As $g'(x) = \cos(x) \forall x \in \mathbb{R}$ and $|\cos(x)| \leq 1 \forall x \in \mathbb{R} \implies |g'(x)| \leq M$ with $M = 1$. Hence g has bounded derivative and the assertion follows from Question 1. □

(c) fg is uniformly continuous,

Proof. FALSE: fg is not uniformly continuous.
As $(fg)'(x) = x\cos(x) + \sin(x)$ we see that $(fg)'$ is not bounded. This does not prove that fg is not uniformly continuous but indicates that the reason for the lack of uniform continuity is that at $x = 2n\pi$, we find that $(fg)'(x) = 2n\pi$ which becomes arbitrarily large as n gets large.
To prove this consider $\delta = \delta_n = \frac{1}{n}$ and show that it is possible to pick $x_n, y_n \in \mathbb{R}$ with $|x_n - y_n| < \delta_n$ but satisfying

$$|x_n \sin(x_n) - y_n \sin(y_n)| \geq 1.$$

We claim that the choice $x_n = 2n\pi$ and $y_n = 2n\pi + \frac{1}{n\pi}$ satisfies the above inequalities. To see this, first note that

$$|x_n - y_n| = \frac{1}{n\pi} < \frac{1}{n}$$

as required. Furthermore, to show that $|x_n \sin(x_n) - y_n \sin(y_n)| \geq 1$, we note that

$$x_n \sin(x_n) = 2n\pi \sin 2n\pi = 0$$

and

$$y_n \sin(y_n) = y_n \sin\left(\frac{1}{n\pi}\right)$$

so that

$$|x_n \sin(x_n) - y_n \sin(y_n)| = y_n \sin\frac{1}{n\pi}.$$

But $\sin(z) > \frac{z}{2}$ for $z \in (0, \pi/2)$ so that $\sin\frac{1}{n\pi} > \frac{1}{2n\pi}$ and therefore

$$|x_n \sin(x_n) - y_n \sin(y_n)| = y_n \sin\left(\frac{1}{n\pi}\right) > \frac{y_n}{2n\pi} > 1$$

as required.

□

(d) The function

$$\begin{cases} \frac{g(x)}{f(x)}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is uniformly continuous.

Proof. TRUE: $h(x)$ is uniformly continuous.
For $x \neq 0$ we use the quotient rule and obtain

$$h'(x) = \left(\frac{g}{f}\right)'(x) = \frac{(x\cos(x) - \sin(x))}{x^2}$$

and for $x = 0$, using L'Hôpital's rule, we get

$$\begin{aligned} h'(0) &= \lim_{x\to 0} \frac{\frac{\sin(x)}{x} - 1}{x} = \lim_{x\to 0} \frac{\sin(x) - x}{x^2} = \lim_{x\to 0} \frac{\cos(x) - 1}{2x} \\ &= \lim_{x\to 0} \frac{-\sin(x)}{2} = 0. \end{aligned}$$

As

$$\lim_{x\to 0} h'(x) = \lim_{x\to 0} \frac{x\cos(x) - \sin(x)}{x^2} = \lim_{x\to 0} \frac{\cos(x) + x\sin(x) - \cos(x)}{2x} = 0 = h'(0)$$

then h' is continuous and hence bounded on $[-L, L]$ for any $L > 0$. Additionally, if $|x| > L$, we estimate

$$|h'(x)| \leq \left|\frac{\cos(x)}{x}\right| + \left|\frac{\sin(x)}{x^2}\right| < \frac{1}{L} + \frac{1}{L^2}.$$

Therefore h' is bounded on $\mathbb{R}$ and the assertion follows from 1. □

4. Let $f : (0, 1) \to \mathbb{R}$ be continuous. Show that

(a) f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists.

Proof. If $A = \lim_{x\to 0} f(x)$ and $B = \lim_{x\to 1} f(x)$ exists then the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} A & x = 0. \\ f(x) & 0 < x < 1, \\ B & x = 1 \end{cases}$$

is continuous on $[0, 1]$ and therefore uniformly continuous on $[0, 1]$. The function f is a restriction of g to the smaller interval $(0, 1)$ and therefore also uniformly continuous. □

(b) If f is uniformly continuous then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists.

Proof. We start by showing that f is bounded. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $|f(x) - f(y))| < \varepsilon$ for any two points $x, y \in (0,1)$ that are less than distance δ apart. Now two arbitrary points $u, v \in (0,1)$ are less than distance one apart and can therefore be connected by a chain of $n = \lfloor \frac{1}{\delta} \rfloor$ points such that two consecutive points are less than distance δ apart. Therefore, $|f(u) - f(v)| < (n+1)\varepsilon$ is finite and f must be bounded. We now show that $\lim_{x\to 0} f(x)$ exists (the case of $x \to 1$ is exactly analogous). As we have established that f is bounded, we know that for $0 < \delta < 1$,

$$a(\delta) = \inf\{f(x) \mid 0 < x < \delta\} \quad \& \quad b(\delta) = \sup\{f(x) \mid 0 < x < \delta\}$$

are well defined, bounded functions of δ. Moreover, $a(\delta)$ increases as $\delta \to 0$ and $b(\delta)$ decreases as $\delta \to 0$. As $a(\delta) \le b(\delta)$ both

$$a = \lim_{\delta\to 0} a(\delta), \quad \& \quad b = \lim_{\delta\to 0} b(\delta)$$

exists. If we can show that $a = b$ then it follows that $\lim_{x\to 0} f(x) = a$. We bound

$$\begin{aligned} b(\delta) - a(\delta) &= \sup\{f(x) \mid 0 < x < \delta\} - \sup\{f(y) \mid 0 < y < \delta\} \\ &= \sup\{f(x) \mid 0 < x < \delta\} + \sup\{-f(y) \mid 0 < y < \delta\} \\ &= \sup\{f(x) - f(y) \mid 0 < y < \delta\} \le \varepsilon, \end{aligned}$$

so that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$b(\delta) \le a(\delta) + \varepsilon.$$

But this implies $b \le a$ where the equality follows. □

5. Show that the following functions are uniformly continuous by directly verifying the ε -δ definition

(a) $h(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$,

Proof. Consider

$$\begin{aligned} |h(x) - h(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &= \left|\frac{x-y}{xy}\right| \end{aligned}$$

Since $x, y > \frac{1}{2} \implies \left|\frac{1}{xy}\right| < 4$ so that

$$\begin{aligned} |h(x) - h(y)| &= \left|\frac{1}{x} - \frac{1}{y}\right| \\ &\le 4|x - y| \end{aligned}$$

Therefore we can choose $\delta = \frac{\varepsilon}{4}$. □

(b) $h(x) = \frac{x}{x+1}$ on $[0, 2]$.

Proof.

$$\begin{aligned}
|h(x) - h(y)| &= \left| \frac{x}{x+1} - \frac{y}{y+1} \right| \\
&\leq \left| \frac{x(y+1) - y(x+1)}{(x+1)(y+1)} \right| \\
&= \left| \frac{x - y}{(x+1)(y+1)} \right|
\end{aligned}$$

Now as $0 < x, y < 2 \implies 1 < x+1, y+1 < 3$ or $|x+1|, |y+1| > 1$ so that

$$\begin{aligned}
|h(x) - h(y)| &= \left| \frac{x}{x+1} - \frac{y}{y+1} \right| \\
&\leq |x - y|
\end{aligned}$$

Therefore we can choose $\delta = \varepsilon$.

$\square$

6. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $c \in \mathbb{R}$.

(a) Given a partition P of $[a, b]$, show that

$$U(cf, P) - L(cf, P) \leq |c|(U(f, P) - L(f, P)).$$

Proof. For $c \geq 0$ we have

$$\sup_{x \in I_i} cf(x) = c \sup_{x \in I_i} f(x), \quad \& \quad \inf_{x \in I_i} cf(x) = c \inf_{x \in I_i} f(x),$$

so that

$$\sup_{x \in I_i} cf(x) - \inf_{x \in I_i} cf(x) = c \left(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \right).$$

If $c \leq 0$, we have instead

$$\sup_{x \in I_i} cf(x) = c \inf_{x \in I_i} f(x), \quad \& \quad \inf_{x \in I_i} cf(x) = c \sup_{x \in I_i} f(x)$$

so that

$$\sup_{x \in I_i} cf(x) - \inf_{x \in I_i} cf(x) = -c \left(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \right).$$

Taken together, we get

$$\sup_{x \in I_i} cf(x) - \inf_{x \in I_i} cf(x) = |c| \left(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \right).$$

Multiplying by $\triangle x_i$ and summing over all i we get the desired result. $\square$

(b) Show that cf is integrable and that

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

Proof. If $U(f,P) - L(f,P) \leq \varepsilon$ for some $\varepsilon > 0$ then also

$$U(cf,P) - L(cf,P) \leq |c|(U(f,P) - L(f,P)) \leq |c|\varepsilon.$$

By Riemann's integrability criterion, cf is integrable. Finally for $c \geq 0$, we have

$$L(cf,P) = cL(f,P) \leq c\int_a^b f(x)dx \leq cU(f,P) = U(cf,P)$$

and for $c \leq 0$ we have

$$L(cf,P) = cU(f,P) \leq c\int_a^b f(x)dx \leq cL(f,P) = U(f,P)$$

so that in both cases

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

follows. □

(c) Let $\alpha \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha, & x \in \left\{\frac{1}{k} \mid k \in \mathbb{N}\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

For which values of α is f Riemann integrable? If f is Riemann integrable what is the value of $\int_0^1 f(x)dx$?

Proof. If $\alpha < 0$ then f is unbounded and therefore not Riemann integrable. Now consider the case where $\alpha \geq 0$. Then f is bounded by 1.
As f is zero on all irrational numbers $L(f,P) = 0$ for all $P \in \mathcal{P}$, and thus

$$\underline{\int_0^1} f(x)dx = 0.$$

Consider the partition on $[0,1]$ by

$$P = \left\{0, \frac{n}{n^2}, \frac{(n+1)}{n^2}, \cdots, \frac{(n^2-1)}{n^2}, \frac{n^2}{n^2}\right\}$$

into one interval of width $\frac{1}{n}$ and $n^2 - n$ intervals of width $\frac{1}{n^2}$.

For $x \geq \frac{1}{n}$, $f(x)$ is non-zero at precisely n points, so that $\sup_{x \in I_i} f(x)$ is non-zero on the left-most interval of width $\frac{1}{n}$ and at most $2n$ intervals of width $\frac{1}{n^2}$. Thus

$$U(f, P_n) \leq \frac{1}{n} + 2n\frac{1}{n^2} = \frac{3}{n}$$

so that f is Riemann integrable and $\int_0^1 f(x)dx = 0$. □