MATH 5105 Differential and Integral Analysis: Exercise Sheet 6

Coursework Exercise

1. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with bounded derivative. Show that f is uniformly continuous.

2. Consider $f(x) = \frac{1}{x^2}$ on $[a, \infty)$ for $a > 0$. Show that f is uniformly continuous.

Problems

3. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x, g : \mathbb{R} \to \mathbb{R}, g(x) = \sin(x)$. Prove or disprove the following statements

   (a) f is uniformly continuous,

   (b) g is uniformly continuous,

   (c) fg is uniformly continuous,

   (d) The function

   $$\begin{cases} \frac{g(x)}{f(x)}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

   is uniformly continuous.

4. Let $f : (0, 1) \to \mathbb{R}$ be continuous. Show that

   (a) f is uniformly continuous if $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists.

   (b) If f is uniformly continuous then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exists.

5. Show that the following functions are uniformly continuous by directly verifying the ε -δ definition

   (a) $h(x) = \frac{1}{x}$ on $[\frac{1}{2}, \infty)$,

   (b) $h(x) = \frac{x}{x+1}$ on $[0, 2]$.

6. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $c \in \mathbb{R}$.

   (a) Given a partition P of $[a, b]$, show that

$$U(cf, P) - L(cf, P) \leq |c|(U(f, P) - L(f, P)).$$

   (b) Show that cf is integrable and that

$$\int_a^b cf(x)dx = c \int_a^b f(x)dx.$$

7. Let $\alpha \in \mathbb{R}$ and $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^\alpha, & x \in \left\{ \frac{1}{k} \mid k \in \mathbb{N} \right\}, \\ 0, & \text{otherwise.} \end{cases}$$

   For which values of α is f Riemann integrable? If f is Riemann integrable what is the value of $\int_0^1 f(x)dx$?