MATH 5105 Differential and Integral Analysis
Exercise Sheet 5

Classwork Exercise

1. If P is a partition and $P' \supset P$ is a refinement then we have

$$U(f, P') \leq U(f, P).$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ and $P' = \{x_0, x_1, x_2, x_3, \ldots, x_{j-1}, \gamma, x_j, \ldots, x_n\}$. Then we compute

$$\begin{aligned}
U(f, P) &= \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) \\
&= \sum_{i=1, i\neq j}^{n} \sup_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) + \sup_{[x_{j-1}, x_j]} f(x)(x_j - x_{j-1}) \\
&\geq \sum_{i=1, i\neq j}^{n} \sup_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) + \sup_{[x_{j-1}, \gamma]} f(x)(\gamma - x_{j-1}) + \sup_{[\gamma, x_j]} f(x)(x_j - \gamma) \\
&= U(f, P').
\end{aligned}$$

Now if P' differs from P by m points, repeat the above argument m times. $\square$

Problems Exercise

2. Let $f(x) = \exp(\sqrt{x}), g(x) = \sin(\pi x)$ and $P = \{0, 1, 4, 9\}$.

   (a) Find the upper and lower sums $U(f, P)$ and $L(f, P)$ of f for the partition P. Use these sums to give bounds for $\int_0^9 f(x)dx$.

   Proof. Recall that $I_i = [x_{i-1}, x_i], \triangle x_i = x_i - x_{i-1}$, and that $M_i = \sup_{x\in I_i} f(x), m_i = \inf_{x\in I_i} f(x)$. We have

$$\begin{aligned}
&I_1 = [0, 1], \quad \triangle_1 = 1, \quad M_1 = \exp(1), \quad m_1 = \exp(0), \\
&I_2 = [1, 4], \quad \triangle_2 = 3, \quad M_2 = \exp(2), \quad m_2 = \exp(1), \\
&I_3 = [4, 9], \quad \triangle_3 = 5, \quad M_3 = \exp(3), \quad m_2 = \exp(2).
\end{aligned}$$

Therefore

$$U(f,P) = \sum_{i=1}^{3} M_i \triangle x_i = 1\exp(1) + 3\exp(2) + 5\exp(3),$$

$$L(f,P) = \sum_{i=1}^{3} m_i \triangle x_i = 1\exp(0) + 2\exp(1) + 5\exp(2).$$

Hence we have

$$1 + 3e + 5e^2 \leq \int_0^9 f(x)dx \leq e + 3e^2 + 5e^3.$$

(In fact the integral evaluates to $2 + 4e^3 \simeq 82.3$, while the lower and upper sums are approximately 46.1 and 125.3.) □

(b) Find the upper and lower sums $U(g,P)$ and $L(g,P)$ of f for the partition P. Use these sums to give bounds for $\int_0^9 g(x)dx$.

Proof.

$$M_1 = 1, m_1 = 0, M_2 = 1, m_2 = -1, M_3 = 1, m_3 = -1.$$

Therefore

$$U(f,P) = 1 \cdot 1 + 3 \cdot 1 + 5 \cdot 1, \quad L(g,P) = 1 \cdot 0 + 3 \cdot (-1) + 5 \cdot (-1).$$

Hence we have

$$-8 \leq \int_0^9 g(x)dx \leq 9.$$

□

Extra Exercises

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

(a) Given a partition P of $[-1,1]$, what is $L(f,P)$? What is $\underline{\int}_{-1}^{1} f(x)dx$?

Proof. Given a partition P of $[-1,1]$, the function f has infimum 0 in any subinterval. Therefore $L(f,P) = 0$ for any partition of P. Hence

$$\underline{\int}_{-1}^{1} f(x)dx = 0.$$

□

(b) For a fixed $\varepsilon > 0$, find a partition P of $[-1, 1]$ such that $U(f, P) < \varepsilon$. Compute $\overline{\int}_{-1}^{1} f(x)dx$.

Proof. For $0 < \delta < 1$ choose $P = \{-1, -\delta, \delta, 1\}$. On the intervals $[-1, -\delta]$ and $[\delta, 1]$ the function f has maximum value 0. On the interval $[-\delta, \delta]$, it has maximum value 1. Therefore

$$U(f, P) = ((-\delta) - (-1)) \cdot 0 + (\delta - (-\delta)) \cdot 1 + (1 - \delta) \cdot 0 = 2\delta,$$

and we choose $\delta < \frac{\varepsilon}{2} \implies U(f, P) < \varepsilon$.

Hence $\overline{\int}_{-1}^{1} f(x)dx \leq 0$. Together with the previous estimate we have

$$0 = \underline{\int}_{-1}^{1} f(x)dx \leq \overline{\int}_{-1}^{1} f(x)dx \leq 0.$$

so that

$$\underline{\int}_{-1}^{1} f(x) = 0$$

□

(c) Is f integrable on$[-1, 1]$? If so compute its integral.

Proof. As

$$\underline{\int}_{-1}^{1} f(x)dx = \overline{\int}_{-1}^{1} f(x)dx = 0,$$

this implies that f is integrable and $\int_{-1}^{1} f(x)dx = 0$. □

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the equidistant partitions P_n on $[0, 1]$ into n subintervals.

Proof. We have

$$P_n = \{0/n, 1/n, \cdots, n/n\},$$

or $x_1 = i/n$ for $i = 0, c \ldots, n$. Thus $I_i = [(i-1)/n, i/n]$ and $\triangle x_i = 1/n$.

□

(a) Find $U(f, P_n)$. What can you say about $\overline{\int}_{0}^{1} f(x)dx$?

Proof. We have $M_i = \left(\frac{i}{n}\right)^2$ and thus

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i \triangle x_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}. \end{aligned}$$

Hence we have

$$\overline{\int_0^1} f(x)dx \le \frac{1}{3}.$$

□

(b) Find $L(f, P_n)$. What can you say about $\underline{\int_0^1} f(x)dx$?

Proof. Similarly we have $m_i = \left(\frac{i-1}{n}\right)^2$ and thus

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i \triangle x_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right) \\ &= \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}. \end{aligned}$$

Hence

$$\overline{\int_0^1} f(x)dx \ge \frac{1}{3}.$$

□

(c) Is f integrable on $[0,1]$? If so, what is its integral?

Proof. Combining the two estimates above we see that $\int_0^1 x^2 dx$ exists and

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

□

[*You may use the formula* $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$.]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \in \mathbb{Q} \text{ where } p, q \text{ are coprime and } q > 0, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that f is Riemann integrable on $[0,1]$,

Proof. Clearly for any partition P of $[0,1]$ we have that $m_i = 0$ for each i as the irrationals are dense in $[0,1]$. Thus $L(f,P) = 0$
If we can show for any $\varepsilon > 0$ we have that $U(f,P) < \varepsilon$, then it follows that $U(f,P) - L(f,P) < \varepsilon$ and by Riemann's condition that f is Riemann integrable. The key for estimating the upper sum is to observe that there are actually very few points at which $f(x)$ is not small. More precisely, given $\varepsilon' > 0$, there are only finitely many points $x \in [0,1]$ such that $f(x) \geq \varepsilon'$ (only those rational numbers with denominator not exceeding $\frac{1}{\varepsilon'}$, that is

$$N + 1 = |\{x \in [0,1] \mid f(x) > \varepsilon'\}|$$

is finite. Let's call these points $y_0 < y_1 < \cdots < y_N$. We now choose a partition

$$P = \{x_0, x_1, x_2, \cdots, x_{2N+1}\}$$

such that

$$x_0 = y_0 < x_1 < x_2 < y_1 < x_3 < x_4 < y_2 < x_5 < x_6 < y_3 < \cdots < x_{2N} < y_N = x_{2N+1},$$

such that $\triangle_{2j+1} = x_{2j+1} - x_{2j} < \frac{\varepsilon'}{N+1}$ for $j = 0, \cdots, N$. Then we can estimate $M_{2j+1} \leq 1$ for $j = 0, \cdots, N$ and $M_{2j} < \varepsilon'$ for $j = 1, \cdots < N$. Splitting the upper sum $U(f,P)$ into even and odd parts, we estimate

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{2N+1} M_i \triangle_i = \sum_{j=0}^{N} M_{2j+1} \triangle_{2j+1} + \sum_{kj=1}^{N} M_{2j} \triangle_{2j} \\ &< \sum_{j=0}^{N} 1 \cdots \triangle_{2j+1} + \sum_{j=1}^{N} \varepsilon' \triangle_{2j} < \frac{(N+1)\varepsilon'}{(N+1)} + \varepsilon' \cdot 1 = 2\varepsilon'. \end{aligned}$$

Thus by choosing $\varepsilon' = \frac{\varepsilon}{2}$ for a given $\varepsilon > 0, U(f,P) < \varepsilon$. □

(b) Show that $\int_0^1 f(x)dx = 0$,

Proof. As $U(f,P) - L(f,P) < \varepsilon$ for all $\varepsilon > 0$ it follows that $\int_0^1 f(x)dx = 0$. □

(c) Show that f is discontinuous at $x \in \mathbb{Q}$ and continuous if $x \notin \mathbb{Q}$.

Proof. Let $x_0 = p/q$ be an arbitrary rational number with $p \in \mathbb{Z}, q \in \mathbb{N}$ such that p and q are coprime. This shows that $f(x_0) = \frac{1}{q}$. Let$\alpha \in \mathbb{R} \backslash \mathbb{Q}$ be any irrational number and define $x_n = x_0 + \frac{\alpha}{n}$ for all $n \in \mathbb{N}$. All these points x_n

are irrational and hence we have that $f(x_n) = 0$ for all $n \in \mathbb{N}$. This implies that

$$|x_n - x_0| = \frac{\alpha}{n} \quad \& \quad |f(x_0) - f(x_n)| = \frac{1}{q}.$$

Let $\varepsilon = \frac{1}{q}$ and given $\delta > 0$let $n = 1 + \lceil \frac{\alpha}{\delta} \rceil$. For the corresponding x_n, we have

$$|f(x_0) - f(x_n) = \frac{1}{q} \geq \varepsilon$$

and

$$|x_0 - x_n| = \frac{\alpha}{n} = \frac{\alpha}{1 + \lceil \frac{\alpha}{\delta} \rceil} < \frac{\alpha}{\lceil \frac{\alpha}{\delta} \rceil} \leq \delta$$

which shows that f is discontinuous at x_0. □

(d) Show that f is nowhere differentiable.

Proof. At a rational number, this follows since f is discontinuous are rational numbers.
For an irrational number let$\{a_n\}$ be any sequence of irrational numbers. The sequence $f(a_n)$ is identically equal to 0 and hence

$$\left| \lim_{n \to \infty} \frac{f(a_n) - f(x_0)}{a_n - x_0} \right| = 0.$$

Furthermore since the rationals are dense in $\mathbb{R}$, it follows that there exists a sequence of rational numbers $\{b_n\} = \left\{ \frac{k_n}{n} \right\}$ converging to x_0 with $k_n \in \mathbb{Z}$ and $b \in \mathbb{N}$ coprime and

$$\left| \frac{k_n}{n} - x_0 \right| < \frac{1}{\sqrt{5}n^2}.$$

Thus for all n we have

$$\left| \frac{f(b_n) - f(x_0)}{b_n - x_0} \right| > \frac{1/n - 0}{1/(\sqrt{5}n^2)} = \sqrt{5}n \neq 0$$

and hence f is not differentiable at any irrational x_0. □

6. Consider $f(x) = \frac{1}{1+x}$ on the interval $[0, 1]$. For each $n \in \mathbb{N}$, define

$$P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1 \right\}.$$

Calculate $L(f, P_n)$ and $U(f, P_n)$ and deduce that f is Riemann integrable on $[0, 1]$.

Proof. We have that $x_k = \frac{k}{n}$ so that $\triangle x_k = \frac{1}{n}$. As f is decreasing $m_k = \frac{1}{1+x_k}$ and $M_k = \frac{1}{1+x_{k-1}}$. Therefore

$$L(f, P_n) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + k/n} = \sum_{k=1}^{n} \frac{1}{n+k}$$

and

$$U(f, P_n) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + (k-1)/n} = \sum_{k=0}^{n-1} \frac{1}{n+k}.$$

Hence

$$U(f, P_n) - L(f, P_n) = \frac{1}{(n+0)} - \frac{1}{(n+n)} = \frac{1}{2n}.$$

So given any $\varepsilon > 0$, if $n > \frac{1}{2\varepsilon}$ then $U(f, P_n) - L(f, P_n) = \frac{1}{2n} < \varepsilon$. Thus by Riemann's condition f is Riemann integrable on $[0, 1]$. $\square$