MATH 5105 Differential and Integral Analysis: Exercise Sheet 5

The deadline is 5pm on Monday 9th March. Please hand in your individual coursework (solution to the coursework exercises) in the maths department. Coursework will be returned during the exercise class following the deadline. Questions are divided as follows -

- Coursework exercises for handing in every two weeks (solutions will be posted),
- Classwork exercises are basic questions designed to help you understand lecture material (solutions will only be given in tutorial classes),
- Problems are exam level questions which require critical thinking (solutions will be posted).

Coursework Exercise

1. If $f : [a, b] \to \mathbb{R}$ is bounded and increasing then f is integrable.
2. Consider the function $f(x) = \sqrt{x}$ on $[0, 1]$.

   The function is increasing and hence Riemann integrable. Compute the lower sum for the partition

   $$P_n = \{0 = x_0, \cdots, x_k = \frac{k^2}{n^2}, \cdots, x_n = 1\}.$$

   You may use $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Classwork Exercise

3. If P is a partition and $P' \supset P$ is a refinement then we have

   $$U(f, P') \leq U(f, P).$$

Problems

4. Let $f(x) = \exp(\sqrt{x}), g(x) = \sin(\pi x)$ and $P = \{0, 1, 4, 9\}$.

   (a) Finder the upper and lower sums $U(f, P)$ and $L(f, P)$ of f for the partition P. Use these sums to give bounds for $\int_0^9 f(x)dx$.

   (b) Find the upper and lower sums $U(g, P)$ and $L(g, P)$ of g for the partition P. Use these sums to give bounds for $\int_0^9 g(x)dx$.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

   (a) Given a partition P of $[-1, 1]$, what is $L(f, P)$? What is $\underline{\int}_{-1}^{1} f(x)dx$?

   (b) For a fixed $\varepsilon > 0$, find a partition P of $[-1, 1]$ such that $U(f, P) < \varepsilon$. Compute $\overline{\int}_{-1}^{1} f(x)dx$.

   (c) Is f integrable on$[-1, 1]$? If so compute its integral.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Consider the equidistant partitions P_n on $[0, 1]$ into n subintervals.

   (a) Find $U(f, P_n)$. What can you say about $\overline{\int}_{0}^{1} f(x)dx$?

   (b) Find $L(f, P_n)$. What can you say about $\underline{\int}_{0}^{1} f(x)dx$?

   (c) Is f integrable on $[0, 1]$? If so, what is its integral?

   [*You may use the formula* $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1)$.]

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q}, \in \mathbb{Q} \text{ where } p, q \text{ are coprime and } q > 0, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

   (a) Prove that f is Riemann integrable on $[0, 1]$,

   (b) Show that $\int_0^1 f(x)dx = 0$,

   (c) Show that f is discontinuous at $x \in \mathbb{Q}$ and continuous if $x \notin \mathbb{Q}$.

   (d) Show that f is nowhere differentiable.

8. Consider $f(x) = \frac{1}{1+x}$ on the interval $[0, 1]$. For each $n \in \mathbb{N}$, define

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}.$$

   Calculate $L(f, P_n)$ and $U(f, P_n)$ and deduce that f is Riemann integrable on $[0, 1]$.