MATH 5105 Differential and Integral Analysis : Exercise Sheet 4

Classroom Exercise

1. Let the function $f : (0, \pi) \to \mathbb{R}$ be given by $f(x) = \cos(x)$.

   (a) Show that f is invertible and show that the inverse $g(y) = f^{-1}(y)$ is differentiable.

   Proof. The image $f((0, \pi))$ is the interval $(-1, 1)$ so $f = g^{-1}$ is defined on this set. As $f'(x) = -\sin(x) < 0$ for $x \in (0, \pi)$, f is strictly decreasing and hence by the inverse function theorem, invertible with differentiable inverse $g : (-1, 1) \to \mathbb{R}$.

   $\square$

   (b) Find the derivative $g'(y)$.

   Proof. The derivative is given by

   $$g'(x) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

   and note that

   $$f'(x) = -\sin(x) = -\sqrt{1 - \cos^2(x)}$$

   so that

   $$g'(x) = -\frac{1}{\sqrt{1 - x^2}}.$$

   $\square$

   (c) Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g and the remainder term in Lagrangian form.

Proof. We have $g(0) = \frac{\pi}{2}.g'(0) = -1$ so that $T_{1,0}(x) = \frac{\pi}{2} - x$. From $g''(x) = -x(1-x^2)^{-3/2}$ the remainder term in Lagrange form is given by

$$R = \frac{1}{2}g''(c)x^2 = -\frac{cx^2}{2(1-c^2)^{3/2}}.$$

□

(d) Show that for $|y| \leq \frac{1}{2}$,

$$|g(y) - \frac{\pi}{2} + y| \leq \frac{\sqrt{3}}{18}.$$

Proof. By Taylor's Theorem, there exists an $c \in [0, x]$ such that $g(x) = T_{1,0}(x) = \mathbb{R}$. For $x \leq \frac{1}{2}$ we get the explicit estimate

$$\begin{aligned} |g(x) - \frac{\pi}{2} + x| &= |g(x) - T_{1,0}(x) \\ &= |R| \leq \frac{|x^3}{2(1-x^2)^{3/2}} \leq \frac{(1/2)^2}{2-(1-(1/4))^{3/2}} = \frac{1}{6\sqrt{3}} = \frac{\sqrt{4}3}{18} \end{aligned}$$

□

Extra Exercises

2. Suppose that the function f satisfies

$$f'(x) = Kf(x)$$

then $f(x) = C\exp(\alpha x)$ for some C, α.

Proof. Follow the proof in the lecture notes for $\exp(x) \implies$ Show that the solution of the equation above is unique up to constant. Then show that $C\exp(\alpha x)$ satisfies the above equation. □

3. Suppose that the function $f(x)$ satisfies the equation

$$f(x+y) = f(x)f(y).$$

(a) If f is differentiable then either $f(x) = 0$ or $f(x) = e^{ax}$.

Proof. We have that

$$\left.\frac{d}{dy}f(x+y)\right|_{y=0} = f'(x) = f(x)f'(0)$$

Therefore

$$f(x) = Ce^{\alpha x}$$

where $C = f(0)$. Substituting this back into the functional equation for f we find that $C^2 = C$ and hence either $C = 0$ or $C = 1$. $\square$

(b*) (hard) If f is continuous then either $f(x) \equiv 0$ or $f(x) = e^{ax}$.

4. Find the $2n$-th derivative of $g(x) = x^2 \sin(x)$ and $h(x) = x^2 \cos(x)$.

Proof.

$$g^{(2n)}(x) = (-1)^{n-1}[(2n^2 - 2n - x^2)\sin(x) + 4nx\cos(x)]$$

$\square$

5. Show that the Taylor series of a polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

is precisely that polynomial.

Proof.

$$f^{(k)}(0) = k!a_k$$

if $0 \le k \le n$ and 0 otherwise. $\square$

6. Let $f(x)$ have continuous derivative in the interval $[a, b]$ and suppose that $f''(x) \ge 0$ for every value of x. Then if ξ is any point in the interval, the curve nowhere falls below its tangent at the point $x = \xi, y = f(\xi)$.

Proof. The equation of the tangent is given by

$$y(x) = f(\xi) + f'(\xi)(x - \xi)$$

By Taylor's theorem, for some η between ξ and x we get that

$$f(x) - y(x) = f(x) - (f(\xi) + f'(\xi)(x - \xi)) = \frac{1}{2}f''(\eta)(x - \xi)^2 \ge 0.$$

$\square$

7. (a) Expand $(1 + x)^{1/2}$ to two terms about $x = 0$ and estimate the remainder.

Proof.

$$T_1(x;0) = 1 + \frac{1}{2}x$$

and the remainder is given by

$$R_2(x;0) = -\frac{1}{16}\frac{x^2}{(1+\xi)^{3/2}}$$

for some ξ between 0 and x. $\square$

(b) Find the best quadratic approximation to $(1+x)^{1/3}$ in a neighbourhood of 0.

Proof.

$$1 + \frac{1}{3}x - \frac{1}{9}x^2$$

$\square$

(c) Find the best quadratic approximation to $(1+x)^{1/n}$ in a neighbourhood of 0.

Proof.

$$1 + \frac{x}{n} + \frac{1}{2n}\left(\frac{1}{n} - 1\right)x^2.$$

$\square$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and let $M_i = \sup_{x\in\mathbb{R}} |f^{(i)}(x)|$ for $i = 0, 1, 2$. Show that

$$M_1^2 \leq 4M_0M_2.$$

Proof. We apply Taylor's theorem to $f(x)$ about a. That is we get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2!}(x-a)^2$$

where c lies between a and x. If we let $x = a + h$ we get

$$f(a+h) = f(a) + f'(a)h + \frac{f''(c_1)}{2!}h^2.$$

If we replace $a + h$ with $a - h$ we get

$$f(a-h) = f(a) - f'(a)h + \frac{f''(c_2)}{2!}h^2.$$

Subtracting the two, we get

$$f'(a) = \frac{f(a-h) - f(a+h)}{2h} + \frac{(f''(c_1) - f''(c_2))h}{2 \cdot 2!}$$

Hence for any a we have

$$M_1 \leq \frac{M_0}{h} + \frac{M_2}{2}h$$

This is true for any value of h in particular it is true of $h = \sqrt{\frac{2M_0}{M_2}}$. □

9. Find the first six terms of the Taylor series for y in powers of x of the following implicitly defined functions

   (a) $x^2 + y^2 = y, y(0) = 0,$

   Proof.

   $$T_6 y(x) = x^2 + x^4 + 2x^6 + \dots$$

   □

   (b) $x^2 + y^2 = y, y(0) = 1,$

   Proof.

   $$T_6 y(x) = 1 - x^2 - x^4 - 2x^6 \dots$$

   □

   (c) $x^3 + y^3 = 0, y(0) = 1.$

   Proof.

   $$T_6 y(x) = x^3 + x^9 + \dots$$

   □

10. Let $f : (-1, \infty) \to \mathbb{R}, f(x) = \sin(\pi(\sqrt{1+x})$.

    (a) Show that

    $$4(1+x)f''(x) + 2f'(x) + \pi^2 f(x) = 0.$$

    Proof. Computation □

    (b) Show that for all $n \in \mathbb{N}$

    $$4f^{(n+1)}(0) + 2(2n+1)f^{(n+1)}(0) + \pi^2 f^{(n)}(0) = 0.$$

Proof. Use induction. □

(c) Find the Taylor polynomial $T_{4,0}(x)$ for $\sin(\sqrt{1+x})$.

11. Note that we have shown that

$$e = \exp(1) = \sum_{k=1}^{\infty} \frac{1}{k!}.$$

Show that the remainder r_n in

$$n!e = n! \sum_{k=1}^{n} \frac{1}{k!} + r_n$$

cannot be an integer and hence e is irrational.

Proof. We find that

$$r_n = \sum_{k=n+1}^{\infty} \frac{n!}{k!}.$$

We estimate each term as

$$0 < \frac{n!}{k!} = \frac{1}{(n+1)(n+2)\cdots k} < \frac{1}{(n+1)^{k-n}}$$

so that

$$0 < r_n < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n}} = \sum_{l=1}^{\infty} \frac{1}{(n+1)^l} = \frac{1}{n}.$$

Hence $0 < r_n < \frac{1}{n} < 1$ and hence r_n cannot be an integer. □