MATH 5105 Differential and Integral Analysis: Exercise Sheet 4

Class work Exercises

1. Let the function $f : (0, \pi) \to \mathbb{R}$ be given by $f(x) = \cos(x)$.

   (a) Show that f is invertible.

   (b) Show that the inverse $g(y) = f^{-1}(y)$ is differentiable.

   (c) Find the derivative $g'(y)$.

   (d) Compute the Taylor polynomial $T_{1,0}(y)$ about zero of degree one for g and the remainder term in Lagrangian form.

   (e) Show that for $|y| \leq \frac{1}{2}$,

$$|g(y) - \frac{\pi}{2} + y| \leq \frac{\sqrt{3}}{18}.$$

Problems

2. Suppose that the function f satisfies

$$f'(x) = Kf(x)$$

   then $f(x) = C\exp(\alpha x)$ for some α, C.

3. Suppose that the function $f(x)$ satisfies the equation

$$f(x + y) = f(x)f(y).$$

   (a) If f is differentiable then either $f(x) = 0$ or $f(x) = e^{ax}$.

   (b*) (hard) If f is continuous then either $f(x) \equiv 0$ or $f(x) = e^{ax}$.

4. Show that the Taylor series of a polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

   is precisely that polynomial.

5. Let $f(x)$ have continuous derivative in the interval $[a, b]$ and suppose that $f''(x) \geq 0$ for every value of x. Then if ξ is any point in the interval, the curve nowhere falls below its tangent at the point $x = \xi, y = f(\xi)$.

6. (a) Expand $(1 + x)^{1/2}$ to two terms about $x = 0$ and estimate the remainder.
   (b) Find the best linear approximation to $(1 + x)^{1/3}$ in a neighbourhood of 0.
   (c) Find the best quadratic approximation to $(1+x)^{1/n}$ in a neighbourhood of 0.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and let $M_i = \sup_{x\in\mathbb{R}} |f^{(i)}(x)|$ for $i = 0, 1, 2$. Show that

$$M_1^2 \leq 4M_0M_2.$$

8. Find the first six terms of the Taylor series for y in powers of x of the following implicitly defined functions
   (a) $x^2 + y^2 = y, y(0) = 0$,
   (b) $x^2 + y^2 = y, y(0) = 1$,
   (c) $x^3 + y^3 = 0, y(0) = 1$.

9. Note that we have shown that

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Show that the remainder R_n in the form

$$n!e = n! \sum_{k=0}^{n} \frac{1}{k!} + R_n$$

cannot be an integer and hence e is irrational.