MATH 5105 Differential and Integral Analysis : Solution Sheet 3

Coursework Exercises

1. Assume that f is continuous on $[a, b]$ and differentiable on (a, b).

   (a) If $f' \leq 0$ then f is non-increasing (monotone decreasing),

   Proof. Let $x, y \in [a, b]$ such that $x < y$. We want to show $f(x) \geq f(y)$. We apply the mean value theorem on $[x, y]$,

   $$f(y) - f(x) = f'(\xi)(y - x) \leq 0$$

   so $f(y) \leq f(x)$.

   □

   (b) If $f' > 0$ then f is strictly increasing,

   Proof. Let $x, y \in [a, b]$ such that $x < y$. We want to show $f(x) < f(y)$. We apply the mean value theorem on $[x, y]$,

   $$f(y) - f(x) = f'(\xi)(y - x) > 0$$

   so $f(y) > f(x)$. □

   (c) If $f' < 0$ then f is strictly decreasing.

   Proof. Let $x, y \in [a, b]$ such that $x < y$. We want to show $f(x) > f(y)$. We apply the mean value theorem on $[x, y]$,

   $$f(y) - f(x) = f'(\xi)(y - x) < 0$$

   so $f(y) < f(x)$.

   □

Problem

2. Let $g = \arctan$ be the inverse of the function $f(x) = \tan(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Applying the inverse function theorem for one variable, find a formula for the derivative $g'(y)$ in terms of y.

   Proof. As $f(x) = \tan(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = I$ and $f(I) = (-\infty, +\infty) = \mathbb{R}$. We define the inverse as $g(y) = \arctan(y), g : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Note that $f'(x) = \sec^2(x) > 0$ so f is differentiable and strictly increasing. Applying the inverse function theorem we get for $y = f(x)$

   $$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2(x)}.$$

   Now we have the identity $\cos^2(x) + \sin^2(x) = 1$, dividing through by $\cos^2(x) \neq 0$ as $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we get

   $$\tan^2(x) + 1 = \sec^2(x).$$

   Therefore

   $$g'(y) = \frac{1}{1 + \sec^2(x)} = \frac{1}{1 + y^2}.$$

   □

3. (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continuous inverse.

   Proof. We take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. Now f is differentiable with a continuous derivative $f'(x) = 3x^2$ with $f'(0) = 0$. The inverse is given by

   $$f^{-1}(x) = x^{1/3}.$$

   As f is strictly increasing on $\mathbb{R}$ f is bijective and $f(\mathbb{R}) = \mathbb{R}$ implies f is surjective and hence f is bijective.
   As f is differentiable, it is continuous therefore f^{-1} is also continuous. □

   (b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove : if $\lim_{x\to 0} f(x) = 0$ then $\lim_{x\to 0} f'(x) = 0$.

   Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -x$. Then f is differentiable and $f'(x) = -1$ for all x and $\lim_{x\to 0} f(x) = 0$ but $\lim_{x\to 0} f'(x) = -1$. □

4. Using the Intermediate Value Theorem, prove that a continuous function maps intervals to intervals.

Proof. We use the following characterisation of an interval: $I \subseteq \mathbb{R}$ is an interval if and only if for all $x_1, x_2 \in I$ with $x_1 < x_2$ we get

$$x_1 < c < x_2 \implies c \in I.$$

Let $J = f(I)$ with $y_1 < y_2$. Then we need to show that J is an interval i.e. that for all $y_1, y_2 \in J$ with $y_1 < y_2$ then $y_1 < c < y_2 \implies c \in J$.

To this end, let us consider $y_1 < y_2$. Then there exists $x_1 < x_2 \in I$ such that $y_1 = f(x_1), y_2 = f(y_2)$. As $y_1 \neq y_2$ it follows that $x_1 \neq x_2$ so that either $x_1 < x_2$ or $x_2 > x_1$. WLOG consider the case $x_1 < x_2$. By assumption f is a continuous function on I, so it is a continuous function on $[x_1, x_2]$.

Hence by the intermediate value theorem, for all c with $y_1 < c < y_2$ there exists an $a \in [x_1, x_2]$ such that $f(a) = c$. This implies that $c \in J$. □

5. Show that $\sin(x) \leq x$ for all $x \geq 0$.

Proof. Let $g(x) = \sin x - x$. Then $g(0) = 0$ and $g'(x) = \cos x - 1$. As $\cos(x) \leq 1$ we see $g'(x) \leq 0$ that is $g(x)$ is monotone decreasing. Hence

$$\sin x - x = g(x) \leq g(0) \leq 0, \quad \forall x > 0 \implies \sin(x) \leq x.$$

Let f be a differentiable function on $\mathbb{R}$ and let

$$a = \sup\{|f'(x)| \mid x \in \mathbb{R}\} < 1.$$

Let $x_0 \in \mathbb{R}$ and define recursively $s_n = f(s_{n-1}), n \geq 1$. Prove that $\{s_n\}$ is convergent sequence and determine its limit.

Proof. Let $a = \sup\{|f'(x)| \mid x \in \mathbb{R}\} < 1$. By MVT, $\forall x \in (a, b) \implies$

$$|f(b) - f(a)| \leq |f'(\xi)||b - a|$$

or

$$|f(x) - f(y)| \leq a|x - y|.$$

Therefore f is a contractive mapping, that is it shrinks the distance between points. Let $s_n = f(s_{n-1}), s_0 \in \mathbb{R}$. Then we will show that $\{s_n\}$ is a Cauchy sequence. Consider

$$\begin{aligned} |s_{n+1} - s_n| &= |f(s_n) - f(s_{n-1})| \\ &\leq a|s_n - s_{n-1}| \\ &\leq a^n|s_1 - s_0|. \end{aligned}$$

Let $n > m$ then

$$\begin{aligned}
|s_m - s_n| &\leq |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \cdots + s_{n+1} - s_n| \\
&\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \cdots + |s_{n+1} - s_n| \\
&\leq (a^{m-(n+1)} + \cdots + 1)|s_{n+1} - s_n| \\
& a^n(a^{m-(n+1)} + \cdots + 1)|s_1 - s_0| \\
&\leq \frac{a^n}{1-a}.
\end{aligned}$$

Since $a < 1, \lim_{n\to\infty} \frac{a^n}{1-a} = 0$, then there exists $N \mid$

$$|s_m - s_n| < \varepsilon, \quad \forall m, n > N.$$

Therefore $\{s_n\}$ is a Cauchy sequence. As the limit exists, we apply the limit laws to conclude

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} f(s_{n-1}) = f\left(\lim_{n\to\infty} s_{n-1}\right) = f(s).$$

□

□