MATH 5105 Differential and Integral Analysis: Exercise Sheet 3

Classwork Exercises

1. Assume that f is continuous on $[a, b]$ and differentiable on (a, b).

   (a) If $f' \leq 0$ then f is non-increasing (monotone decreasing),

   (b) If $f' > 0$ then f is strictly increasing,

   (c) If $f' < 0$ then f is strictly decreasing.

Problems

2. Let $g = \arctan$ the inverse of the function $f(x) = \tan(x), x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Applying the inverse function theorem for one variable, find a formula for the derivative $g'(y)$ in terms of y.

3. (a) Find a bijective, continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) = 0$ and a continuous inverse.

   (b) Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be differentiable and decreasing. Prove or disprove : if $\lim_{x\to 0} f(x) = 0$ then $\lim_{x\to 0} f'(x) = 0$.

4. Show that $\sin(x) \leq x$ for all $x \geq 0$.

5. (Hard) Using the Intermediate Value Theorem, prove that a continuous function maps intervals to intervals.

6. (Hard) Let f be a differentiable function on $\mathbb{R}$ and let

$$a = \sup\{|f'(x)| \mid x \in \mathbb{R}\} < 1.$$

   Let $x_0 \in \mathbb{R}$ and define recursively $s_n = f(s_{n-1}), n \geq 1$. Prove that $\{s_n\}$ is convergent sequence and determine its limit.