MATH 5105 Differential and Integral Analysis : Solution Sheet 2

Coursework Exercises

1. For the following functions compute directly
$$r(x) = f(x) - f(a) - f'(a)(x-a)$$
and show that $\lim_{x\to a} \frac{r(x)}{x-a} = 0$.

   (a) $f(x) = x^2, a \in \mathbb{R}$,

   Proof.
$$r(x) = x^2 - a^2 - 2a(x-a)$$
   Therefore
$$\begin{aligned}\lim_{x\to a} \frac{r(x)}{x-a} &= \lim_{x\to a} \frac{x^2 - a^2 - 2a(x-a)}{x-a}\\ &= \lim_{x\to a} x + 1 - 2a\\ &= 0.\end{aligned}$$
   □

   (b) $f(x) = \sqrt{x}, a > 0$,

   Proof.
$$r(x) = \sqrt{x} - \sqrt{a} - \frac{1}{2\sqrt{a}}(x-a)$$
   Therefore
$$\begin{aligned}\lim_{x\to a} \frac{r(x)}{x-a} &= \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a} - \frac{1}{2\sqrt{a}}(x-a)}{x-a}\\ &= \lim_{x\to a} \frac{\frac{x-a}{\sqrt{x}+\sqrt{a}} - \frac{1}{2\sqrt{a}}(x-a)}{x-a}\\ &= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} - \frac{1}{2\sqrt{a}}\\ &= 0.\end{aligned}$$

□

(c) $f(x) = x^n, a \in \mathbb{R}, n \in \mathbb{N}$.

Proof.

$$\lim_{x\to a}\frac{r(x)}{x-a} = \lim_{x\to a}\frac{x^n - a^n - na^{n-1}(x-a)}{x-a}$$
$$= \lim_{x\to a}\frac{(x-a)\sum_{k=0}^{n-1} x^k a^{n-1-k} - na^{n-1}(x-a)}{x-a}$$
$$= \lim_{x\to a}\sum_{k=0}^{n-1} x^k a^{n-1-k} - na^{n-1} = 0.$$

□

Problems

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function that satisfies $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Show that

$$|f(x) - f(y)| \leq |x - y|, \quad \forall x, y \in \mathbb{R}.$$

Proof. Without loss of generality, assume that $y > x$. We then apply the Mean Value Theorem on $[x, y]$. We obtain

$$f(y) - f(x) = f'(\xi)(y - x)$$

for some $\xi \in (x, y)$. This implies that

$$|f(x) - f(y)| \leq |f'(\xi)||y - x| \leq |y - x|.$$

□

3. Let f be defined on $\mathbb{R}$ and suppose that

$$|f(x) - f(y)| \leq |x - y|^2 \quad \forall x, y \in \mathbb{R}$$

Show that f is a constant function.

Proof. From the above inequality for all $x, a \in \mathbb{R}, x \neq a$ we have

$$\left|\frac{f(x) - f(a)}{x - a}\right| \leq |x - a|$$

We claim that this shows that f is differentiable at a with $f'(a) = 0$. To see this we note that for any $\varepsilon > 0$ we may choose $\delta = \varepsilon$ and then if $0 < |x - a| < \delta$ then

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| = \left| \frac{f(x) - f(a)}{x - a} \right| \leq |x - a| < \delta = \varepsilon.$$

In other words,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$$

Therefore as $f' = 0$ on a closed interval, $f \equiv const.$ on $\mathbb{R}$. □

Extra Exercises

4. Let, $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g, \quad \& \quad g' = -f$$

Show that between every two zeroes of f there is a zero of g and between every two zeroes of g there is a zero of f.

Proof. Choose $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b) = 0$. As f is differentiable on $\mathbb{R}$, the assumptions of Rolle's theorem are satisfied on $[a, b]$, that is f is continuous on $[a, b]$ and differentiable on (a, b).

Therefore there exists $c \in (a, b)$ such that $f'(c) = 0$. As $f' = g, g(c) = f'(c) = 0$. An analogous argument is valid with f and g exchanged. □

5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable [that is $(f')' = f''$] with

$$f(0) = f'(0) = 0 \quad \& \quad f(1) = 1.$$

Show that there exists a $c \in (0, 1)$ such that $f''(c) > 1$.

Proof. As f is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, 1]$, that is f is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

Therefore there exists $d \in (0, 1)$ such that

$$f'(d) = \frac{f(1) - f(0)}{1 - 0} = 1$$

As f' is differentiable on $\mathbb{R}$, the assumptions of the MVT are satisfied on $[0, d]$. Therefore there exists a $c \in (0, d)$ such that

$$f''(c) = \frac{f'(d) - f'(0)}{d - 0} = \frac{1}{d}.$$

As $d \in (0, 1)$, $\frac{1}{d} > 1$. □

6. Suppose that f is continuous on $[0, 1]$, differentiable on $(0, 1)$ and $f(0) = 0$. Prove that if f' is decreasing on $(0, 1)$ then the function $g : (0, 1) \to \mathbb{R}$ given by $g(x) = \frac{f(x)}{x}$ is decreasing on $(0, 1)$.

   Proof. Since g is differentiable on $(0, 1)$ then it suffices to show that $g'(x) \leq 0$ for all $x \in (0, 1)$. As

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

   we only need to show that $f'(x)x - f(x) \leq 0$. We apply the MVT to f on $[0, x]$, there exists $c \in (0, x)$ such that $f(x) - f(0) = f'(c)(x - 0)$. As f' is decreasing and $c < x$ then $f'(x) \leq f'(c)$. Therefore

$$f(x) = f'(c)x \geq f'(x)x$$

   and hence $f'(x)x - f(x) \leq 0$ for all $x \in (0, 1)$. □