MATH 5105 Differential and Integral Analysis
Exercise Sheet 2

Coursework Exercises

1. For the following functions compute directly

$$r(x) = f(x) - f(a) - f'(a)(x - a)$$

and show that $\lim_{x \to a} \frac{r(x)}{x-a} = 0$.

   (a) $f(x) = x^2, a \in \mathbb{R}$,
   (b) $f(x) = \sqrt{x}, a > 0$,
   (c) $f(x) = x^n, a \in \mathbb{R}, n \in \mathbb{N}$.

Problems

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function that satisfies $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Show that

$$|f(x) - f(y)| \leq |x - y|, \quad \forall x, y \in \mathbb{R}.$$

3. Let f be defined on $\mathbb{R}$ and suppose that

$$|f(x) - f(y)| \leq |x - y|^2 \quad \forall x, y \in \mathbb{R}$$

Show that f is a constant function.

4. Let, $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable with

$$f' = g, \quad \& \quad g' = -f$$

Show that between every two zeroes of f there is a zero of g and between every two zeroes of g there is a zero of f.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable (that is $(f')' = f''$) with

$$f(0) = f'(0) = 0 \quad \& \quad f(1) = 1.$$

Show that there exists a $c \in (0, 1)$ such that $f''(c) > 1$.

6. Suppose that f is continuous on $[0, 1]$, differentiable on $(0, 1)$ and $f(0) = 0$. Prove that if f' is decreasing on $(0, 1)$ then the function $g : (0, 1) \to \mathbb{R}$ given by $g(x) = \frac{f(x)}{x}$ is decreasing on $(0, 1)$.