MATH 5105 Differential and Integral Analysis: Solution Sheet 1

Coursework Exercises

1. Using the definition of continuity, show that the following functions are continuous

   (a) $f(x) = x^2$ at $x = 0$,

   Proof. Let us consider $\left|x^2\right|$. Then if we choose $|x| < \delta$ where $\delta = \sqrt{\varepsilon}$, then we see that $|x|^2 = |x||x| < \delta^2 = \varepsilon$. $\square$

   (b) $f(x) = |x|$ on $\mathbb{R}$,

   Proof. Let $a \in \mathbb{R}$. Then let us consider $||x| - |a||$. We see that by the reverse triangle inequality $||x| - |a|| \leq |x - a|$. Therefore if $|x - a| < \delta$ where we choose $\delta = \varepsilon$ then $||x| - |a|| \leq |x - a| < \delta = \varepsilon$. $\square$

   (c) $f(x) = \frac{1}{x^2}$ on $(0, \infty)$.

   Proof. Let $a \in \mathbb{R}$ where $a > 0$. Then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x^2-a^2|}{|x|^2|a|^2}$. We choose here that $|x - a| < \frac{a}{2}$ so that $|x| > \frac{a}{2}$. Furthermore $|x + a| \leq \frac{5a}{2}$. This then gives

   $$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{a^2}\right| &= \frac{|x^2 - a^2|}{|x|^2|a|^2} = \frac{|x-a||x+a|}{|x|^2|a|^2} \\ &\leq \frac{|x-a||x+a|}{|a|^2} \times \frac{4}{a^2} \leq \frac{4|x-a||x+a|}{a^4} \leq \frac{10|x-a|}{a^3}.\end{aligned}$$

   Therefore we choose $\delta = \min\left\{\frac{a^3\varepsilon}{10}, \frac{a}{2}\right\}$. $\square$

2. Use the definition of derivative to calculate the derivatives of the following functions

   (a) $f(x) = \sqrt{x}$ for $x \in (0, \infty)$, *Proof.* The derivative is given by $f'(x) = \frac{1}{2\sqrt{x}}$. We compute as follows

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

□

(b) $f(x) = (x+2)^2$ for $x \in \mathbb{R}$,

Proof. Computing

$$\lim_{h\to 0}\frac{(x+h+2)^2-(x+2)^2}{h} = \lim_{h\to 0}\frac{2h(x+2)}{h} = 2(x+2).$$

□

(c) $f(x) = x^2\cos(x)$ at $x = 0$,

Proof. Computing we have

$$\begin{aligned}f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0}\frac{h^2\cos(h)-0}{h-0}\\ &= \lim_{h\to 0} h\cos h = 0.\end{aligned}$$

□

Problems

3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x^2}\right), & x \neq 0\\ 0, & x = 0\end{cases}$$

(a) Show that f is differentiable at $x = 0$ and compute $f'(0)$,

Proof. Consider the difference quotient

$$\frac{f(x)-f(0)}{x-0} = \frac{x^2\sin\frac{1}{x^2}-0}{x-0} = x\sin\frac{1}{x^2}.$$

Given any $\varepsilon > 0$, let $\delta = \varepsilon$. Then given any $0 < |x| < \delta$, we have

$$\left| x \sin \frac{1}{x^2} - 0 \right| = |x| \left| \sin \frac{1}{x^2} \right| \leq |x| \leq \delta$$

as $|\sin y| \leq 1, \forall y \in \mathbb{R}$. Hence

$$\lim_{x \to 0} x \sin \frac{1}{x^2} = 0$$

Therefore f is differentiable at zero with $f'(0) = 0$. □

(b) Find $f'(x)$ for $x \neq 0$ (given that $\frac{d}{dx} \sin x = \cos x$),

$$\begin{aligned} f'(x) &= 2x \sin \left(\frac{1}{x^2} \right) - \frac{2}{x^3} x^2 \cos \left(\frac{1}{x^2} \right) \\ &= 2x \sin \left(\frac{1}{x} \right) - \frac{2}{x} \cos \left(\frac{1}{x^2} \right). \end{aligned}$$

(c) Show that $f'(x)$ is not continuous at $x = 0$. Now we see that the first term satisfies $\lim_{x \to 0} 2x \sin \left(\frac{1}{x^2} \right) = 0$ but

$$-\frac{2}{x} \cos \left(\frac{1}{x^2} \right)$$

has no limit as $x \to 0$ hence $f(x)$ has no limit as $x \to 0$ so f is not continuous.

4. Let $f : [-1, 1] \to \mathbb{R}$ be continuous on $[-1, 1]$. Assume that f is differentiable at $x = 0$ and $f(0) = 0$. Consider the function

$$g(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

(a) Show that g is continuous at $x = 0$,

Proof. A function g is continuous at a if $\lim_{x \to a} g(x) = g(a)$. Here we let $a = 0$ and

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = g(0)$$

Hence g is continuous at 0. □

(b) Is g continuous for $x \neq 0$?

Proof. g is clearly continuous if $x \neq 0$ as g is the quotient of continuous functions and the numerator is not zero for $x \neq 0$. □

(c) Deduce that there is some number M such that

$$\frac{f(x)}{x} \leq M, \quad \forall x \in [-1,1]\backslash\{0\}.$$

Proof. Since g is continuous on a closed bounded interval, this shows that g is bounded that on $[-1,1] \implies \exists M > 0 \mid |g(x)| \leq M \forall x \in [-1,1]$. □

5. Give an example of a function f that is differentiable on (a,b) but that can not be made differentiable on $[a,b]$ by any definition of $f(a)$ or $f(b)$. Can you give an example where f is bounded?

Proof. There are many possible examples. For example consider $f(x) = \frac{1}{(x-a)(x-b)}$. f is clearly differentiable and continuous on (a,b) but cannot be made continuous at $x = a$ or $x = b$ by any definition of $f(a), f(b)$.

We consider a convolution of the above function with a bounded function

$$g(x) = \sin\left(\frac{1}{(x-a)(x-b)}\right),$$

g is clearly differentiable (and continuous) on (a,b) but cannot be made continuous at $x = a$ or $x = b$ by any definition of $g(a), g(b)$. □

6. Let $f(x) = x\sin\left(\frac{1}{x}\right), x \neq 0, \quad f(0) = 0$.

(a) Show that f is continuous at $x = 0$.

Proof. We will show that f is continuous at $x = 0$ by showing that

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

Given $\varepsilon > 0$ we choose $\delta = \varepsilon$ so that if $0 < |x| < \delta = \varepsilon$ then

$$\left|x\sin\left(\frac{1}{x}\right) - 0\right| \leq |x|\left|\sin\left(\frac{1}{x}\right)\right| = |x| < \delta = \varepsilon.$$

□

(b) Is f differentiable at $x = 0$? Justify any answer.

Proof. f is not differentiable because if we compute the difference quotient

$$\frac{f(x) - f(0)}{x - 0} = \sin\left(\frac{1}{x}\right),$$

this has no limit as $x \to 0$. □