

MATH 5105 Differential and Integral Analysis

Exercise Sheet 1

- Coursework exercises are basic questions designed to help you understand lecture material. These questions should be able to be completed without reference to solutions.
- Problems are exam level questions which require critical thinking.

Coursework Exercises

1. Using the definition of continuity, show that the following functions are continuous
 - (a) $f(x) = x^2$ at $x = 0$,
 - (b) $f(x) = |x|$ on \mathbb{R} ,
 - (c) $f(x) = \frac{1}{x^2}$ on $(0, \infty)$.
2. Use the definition of derivative to calculate the derivatives of the following functions
 - (a) $f(x) = \sqrt{x}$ for $x \in (0, \infty)$,
 - (b) $f(x) = (x + 2)^2$ for $x \in \mathbb{R}$,
 - (c) $f(x) = x^2 \cos(x)$ at $x = 0$.

Problems

3. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- (a) Show that f is differentiable at $x = 0$ and compute $f'(0)$,
- (b) Find $f'(x)$ for $x \neq 0$ (given that $\frac{d}{dx} \sin x = \cos x$),
- (c) Show that $f'(x)$ is not continuous at $x = 0$.

4. Let $f : [-1, 1] \rightarrow \mathbb{R}$ be continuous on $[-1, 1]$. Assume that f is differentiable at $x = 0$ and $f(0) = 0$. Consider the function

$$g(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

- (a) Show that g is continuous at $x \neq 0$,
- (b) Is g continuous for $x = 0$?
- (c) Deduce that there is some number M such that

$$\frac{f(x)}{x} \leq M, \quad \forall x \in [-1, 1] \setminus \{0\}.$$

5. Give an example of a function f that is differentiable on (a, b) but that can not be made differentiable on $[a, b]$ by any definition of $f(a)$ or $f(b)$. Can you give an example where f is bounded?
6. Let $f(x) = x \sin\left(\frac{1}{x}\right)$, $x \neq 0$, $f(0) = 0$.
- (a) Show that f is continuous at $x = 0$.
 - (b) Is f differentiable at $x = 0$? Justify any answer.