

MTH5130 Mock Exam Paper

9th January 2024

Q1

1. Find all integers satisfying $10x \equiv 511 \pmod{841}$. Show your working. **[4]**
2. Find the last two digits of 2^{2021} . Show your working. **[8]**
3. Find all integers of order 6 mod 13. Moreover, find all primitive roots mod 13. Show your working in both cases. **[8]**

Q2

1. Deduce that 143 is not a prime number from the congruence $3^{143} \equiv 126 \pmod{143}$. State clearly any result you are using from lectures. **[3]**
2. Let p be a prime number and let z be a primitive root mod p . Prove that

$$1, z, z^2, \dots, z^{p-2}$$

are all distinct mod p . [Hint: z is invertible mod p , i.e. for any integers a and b , if $za \equiv zb \pmod{p}$, then $a \equiv b \pmod{p}$, and z has order $p - 1$] **[9]**

3. Assume that 741 and 9283 are prime numbers. Using the properties of Legendre symbol, compute the Legendre symbol $\left(\frac{741}{9283}\right)$. Justify your answer. **[6]**

Q3

Which of the following congruences are soluble? If soluble, find a positive integer solution less than 47; if insoluble, explain why.

(i) $x^2 \equiv 41 \pmod{47}$. **[4]**

(ii) $3x^2 \equiv 32 \pmod{47}$. **[8]**

Q4

1. Compute the continued fraction expression for $\sqrt{23}$. Show your working. **[4]**
2. Compute the convergents $\frac{s_1}{t_1}, \frac{s_2}{t_2}, \frac{s_3}{t_3}$ to $\sqrt{23}$. Show your working. **[4]**
3. (\geq Week 9) By working out the second smallest positive solution to the equation $x^2 - 23y^2 = 1$, compute the convergent $\frac{s_7}{t_7}$. **[10]**

Q5

1. Using that 137 is a prime number, find all solutions to

$$x^2 \equiv -1 \pmod{137}$$

satisfying $1 \leq x \leq 137$. Show your working. **[9]**

2. (\geq Week 10) Using (1), write 137 as a sum of two squares. Show your working. State clearly any results you are using from lectures. **[9]**

Q6

1. Describe the units in the ring of integers in $\mathbb{Q}(\sqrt{75})$. **[14]**.