Exercise sheet 11 solutions

1. **Non-numerical parts**

   This formula is given in the question (was explained briefly where it came from in one of the lectures):

   $$p(y \mid M_2) = \frac{\sigma_1\, p(y \mid M_1)}{\sigma_0\, exp\left(-(\mu_1 - \mu_0)^2/(2\sigma_1^2)\right)}.$$

   Part (a)
   The Bayes factor B_{12} follows just by rearranging the preceding formula

   $$B_{12} = \frac{p(y \mid M_1)}{p(y \mid M_2)} = \frac{\sigma_0}{\sigma_1}\, exp\left(-\frac{(\mu_1 - \mu_0)^2}{2\sigma_1^2}\right).$$

   Part (b)
   The posterior parameters in model M_2 are

   $$\mu_1 = \frac{\mu_0/\sigma_0^2 + n\bar{y}/\sigma^2}{1/\sigma_0^2 + n/\sigma^2},\ \sigma_1^2 = \frac{1}{1/\sigma_0^2 + n/\sigma^2}.$$

   For large enough σ_0, we can ignore the $\frac{1}{\sigma_0^2}$ terms, and so $\mu_1 \approx \frac{n\bar{y}/\sigma^2}{n/\sigma^2} = \bar{y}$ and $\sigma_1^2 \approx \frac{\sigma^2}{n}$.
   Hence

   $$B_{12} \approx \frac{\sqrt{n}\sigma_0}{\sigma}\, exp\left(-\frac{n(\bar{y} - \mu_0)^2}{2\sigma^2}\right).$$

   Part (f)
   Once the prior standard deviation σ_0 reaches a large enough value (compared to $\sigma/\sqrt{n}$, which determines the scale of the likelihood function), then increasing σ_0 further hardly changes the posterior mean and standard deviation for μ in model M_2. This is because for a large enough value of σ_0, the posterior density is approximately proportional to the likelihood.

   However the posterior model probabilities do change as σ_0 increases. This is due to the Bayes factor increasing approximately proportional to σ_0 for large enough σ_0, so here the Bayes factor is approximately multiplied by 10 when σ_0 changes from 10 to 100.

2. For model M_2, the likelihood for the Poisson model with mean λ is

   $$p(y \mid \lambda, M_2) = \frac{\lambda^S e^{-n\lambda}}{\prod_{i=1}^n y_i!},\ \text{where } S = \sum_{i=1}^n y_i.$$

   The prior distribution for model M_2 is

   $$p(\lambda \mid M_2) = \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}.$$

For general data

$$p(y \mid M_1) = \frac{e^{-n}}{\prod_{i=1}^{n} y_i!},$$

the likelihood function with $\lambda = 1$.

$$\begin{aligned} p(y \mid M_2) &= \int_0^\infty p(\lambda \mid M_2)\, p(y \mid \lambda, M_2)\, d\lambda \\ &= \int_0^\infty \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)} \, \frac{\lambda^S e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}\, d\lambda \end{aligned}$$

(a) Here $S = 0$ and $\alpha = 1$, $\beta = 1$. In this case

$$p(y \mid M_1) = e^{-n}.$$

$$p(y \mid M_2) = \int_0^\infty e^{-\lambda}\, e^{-n\lambda}\, d\lambda = \int_0^\infty e^{-(n+1)\lambda}\, d\lambda = \frac{1}{n+1}.$$

Hence the Bayes factor is

$$B_{12} = \frac{p(y \mid M_1)}{p(y \mid M_2)} = (n+1)\, e^{-n}.$$

For $n = 10$, $B_{12} = 0.000499$.
The posterior probability of model M_1 is

$$p(M_1 \mid y) = \frac{B_{12}}{1 + B_{12}} = 0.000499.$$

(b) For general data and $\alpha = 1$, $\beta = 1$,

$$p(y \mid M_2) = \int_0^\infty e^{-\lambda} \, \frac{\lambda^S e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}\, d\lambda = \int_0^\infty \frac{\lambda^S e^{-(n+1)\lambda}}{\prod_{i=1}^{n} y_i!}\, d\lambda$$

Put $w = (n+1)\lambda$, $\dfrac{dw}{d\lambda} = n+1$.

$$\begin{aligned} p(y \mid M_2) &= \int_0^\infty \frac{w^S e^{-w}}{(n+1)^{S+1} \prod_{i=1}^{n} y_i!}\, dw = \frac{\Gamma(S+1)}{(n+1)^{S+1} \prod_{i=1}^{n} y_i!} \\ &= \frac{S!}{(n+1)^{S+1} \prod_{i=1}^{n} y_i!} \end{aligned}$$

Hence the Bayes factor is

$$B_{12} = \frac{(n+1)^{S+1} e^{-n}}{S!}.$$