PS9 Q7:

seperation of variables gives $u(x,t) = X(x)T(t)$

and $X\dot{T} = k X''T$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} = -\lambda$$

with $\lambda \geq 0$. (proof of $\lambda \geq 0$ is similar to the previous problems using integration by parts)

The eigenvalue problem is then

$$\begin{cases} X'' = -\lambda X \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{cases}$$

The general solution for $X(x)$ is

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

and so

$$X'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

using $X(-L) = X(L)$, we have

$$C_1\cos(-\sqrt{\lambda}L) + C_2\sin(-\sqrt{\lambda}L) = C_1\cos(\sqrt{\lambda}L) + C_2\sin(\sqrt{\lambda}L)$$

$$C_1[\cos(-\sqrt{\lambda}L) - \cos(\sqrt{\lambda}L)] = C_2[\sin(\sqrt{\lambda}L) - \sin(-\sqrt{\lambda}L)] \quad (1)$$

using $X'(-L) = X'(L)$, we have

$$-C_1\sqrt{\lambda}\sin(-\sqrt{\lambda}L) + C_2\sqrt{\lambda}\cos(-\sqrt{\lambda}L) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

$$C_2\sqrt{\lambda}\left[\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)\right]=C_1\sqrt{\lambda}\left[\sin(-\sqrt{\lambda}L)-\sin(\sqrt{\lambda}L)\right] \quad (2)$$

There are 2 cases:

Case 1:

If $\lambda=0$, then we have $X(x)=\text{const}$; $\lambda_0=0$

Case 2:

If $\lambda\neq 0$, we have 2 sub cases:

sub case 2.1: If neither $[\sin(\sqrt{\lambda}L)+\sin(-\sqrt{\lambda}L)]$ nor $[\cos(\sqrt{\lambda}L)-\cos(-\sqrt{\lambda}L)]$ is zero,

by (1), $$\frac{c_1}{c_2}=\frac{\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)}{\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)}$$

by (2), $$-\frac{c_2}{c_1}=\frac{\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)}{\cos(-\sqrt{\lambda}L)-\cos(\sqrt{\lambda}L)}$$

so $$\frac{c_1}{c_2}=\frac{-c_2}{c_1}$$

multiply both sides by a_1a_2, get

$$c_1^2=-c_2^2$$

$$c_1^2+c_2^2=0$$

So $c_1=0$ and $c_2=0$, this is the trivial solution $X\equiv 0$.

sub case 2.1: either $\sin(\sqrt{\lambda}L)-\sin(-\sqrt{\lambda}L)=0$

or $\cos(\sqrt{\lambda}L)-\cos(-\sqrt{\lambda}L)=0$, If $\sin(\sqrt{\lambda}L) - \sin(-\sqrt{\lambda}L) = 0$

then $\sqrt{\lambda}L - (-\sqrt{\lambda}L) = 2n\pi, \quad n=1,2,\cdots$

$$2\sqrt{\lambda}L = 2n\pi$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n=1,2,\cdots$$

If $\cos(\sqrt{\lambda}L) - \cos(-\sqrt{\lambda}L) = 0$

then $\sqrt{\lambda}L - (-\sqrt{\lambda}L) = 2n\pi, \quad n=1,2,\cdots$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

both give $X_n(x) = a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$.

Knowing λ_n, we have

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt}$$

The general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2}{L^2}kt}$$

using the initial condition, we have

when $t=0$

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right]$$

By the formula of Fourier series coefficients,

we get

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

8. Notice

$$f(x) = \sin^2\left(\frac{\pi x}{L}\right) = \frac{1-\cos\frac{2\pi x}{L}}{2}$$

$$= \frac{1}{2} - \frac{1}{2}\cos\frac{2\pi x}{L}$$

By the orthogonality of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$, we know that

$a_0 = \frac{1}{2}$,

$a_n = 0$ except for $n=2$

$a_2 = -\frac{1}{2}$

$b_n = 0$ for all n.

so plugging into the general solutions, we get.

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\cos\frac{2\pi x}{L}\, e^{-\frac{4\pi^2}{L^2}kt}$$