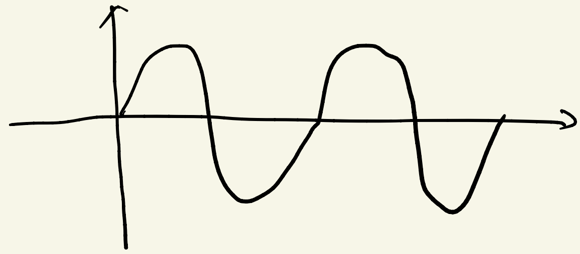
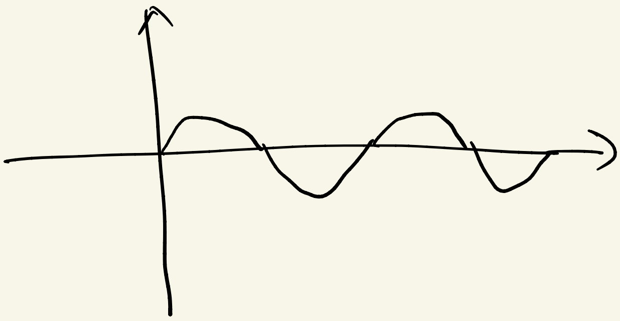
Selected solutions to PS 11

1. The rough behaviour of u over time is as follows

$t=0$

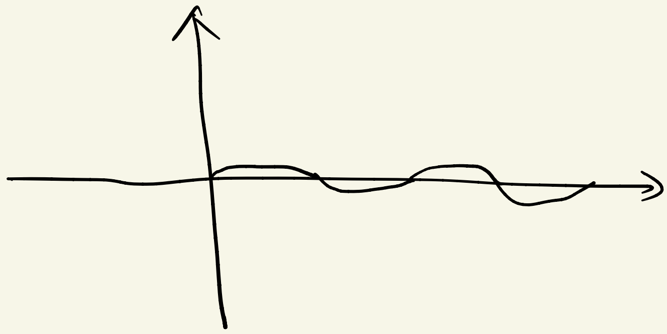


$t=1$



⋮

t large



2. $$\frac{d}{dt}\int_0^L u(x,t)\,dx$$

$$= \int_0^L \frac{d}{dt}\left[u(x,t)\right]dx$$

$$= \int_0^L u_t(x,t)\,dx$$

$$= \int_0^L k u_{xx}(x,t)\,dx \quad \leftarrow \text{by the PDE}$$

$$= k u_x \Big|_0^L$$

$$= k u_x(L,t) - k u_x(0,t)$$ ← by the Neumann boundary condition

$$= 0 - 0$$

$$= 0$$

so $\int_0^L u(x,t)\,dx$ is a conserved quantity.

3. If u does not depend on t, the PDE becomes

$$0 = u_t = k u_{xx}$$

so

$$u_{xx} = 0$$

$$u_x = c$$

$$u = cx + d \quad \text{a linear function}$$

If the temperature distribution does not change with time, it has to be a linear function.

4. Let us consider $u(x,t) = e^{-bt} v(x,t)$

namely $v(x,t) = e^{bt} u(x,t)$

we have $v_x(x,t) = e^{bt} u_x(x,t)$

$$v_{xx}(x,t) = e^{bt} u_{xx}(x,t)$$

and $V_t(x,t) = be^{bt} u(x,t) + e^{bt} u_t(x,t)$

So we get

$$V_t - k V_{xx} = be^{bt} \cdot u + e^{bt} \cdot u_t - ke^{bt} \cdot u_{xx}$$

$$= e^{bt} \cdot [u_t - k u_{xx} + bu]$$

by the PDE satisfied by u. →

$$= e^{bt} \cdot 0$$

$$= 0$$

So V satisfied the equation

$$\begin{cases} V_t - k V_{xx} = 0 \\ V(x,0) = e^{0 \cdot t} u(x,0) = e \cdot f(x) \end{cases}$$

By Fourier-Poisson formula, we get

$$V(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot e \cdot f(y)\, dy$$

and so

$$u(x,t) = e^{-bt} V(x,t)$$

$$= \frac{e^{-bt}}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot e \cdot f(y)\, dy$$