Random Processes – January 2023 Exam Solutions

You are strongly encouraged to attempt the paper and to check your solutions without looking at these solutions. However, if you get really stuck on a question then it may be useful to have these. For some questions there may be several equally valid answers or aproaches so your answer might be correct even if it doesn't look exactly the same as mine.

1.

(a) The only way to go from state 4 to state 2 in two steps is to follow the path 4 to 3 to 2. So

$$\mathbb{P}(X_{10} = 2 \mid X_8 = 4) = p_{4,3}p_{3,2} = \tfrac{1}{2} \times \tfrac{1}{6} = \tfrac{1}{12}.$$

(b) Let T be the time of absorption. Define $a_i = \mathbb{P}(X_T = 5 \mid X_0 = i)$. The question asks for a_3. By first step analysis (conditioning on the first step) we get the equations:

$$\begin{aligned} a_2 &= \tfrac{1}{4} + \tfrac{1}{2}a_3 \\ a_3 &= \tfrac{1}{2} + \tfrac{1}{6}a_2 + \tfrac{1}{6}a_3 + \tfrac{1}{6}a_4 \\ a_4 &= \tfrac{1}{2}a_3 + \tfrac{1}{2}a_4 \end{aligned}$$

From the last equation $a_3 = a_4$. Substituting this into the second equation we get

$$a_3 = \tfrac{1}{2} + \tfrac{1}{6}a_2 + \tfrac{1}{3}a_3$$

and so

$$a_2 = 4a_3 - 3$$

Now using the first equation

$$4a_3 - 3 = \tfrac{1}{4} + \tfrac{1}{2}a_3$$

which gives

$$\tfrac{7}{2}a_3 = \tfrac{13}{4}$$

and so $a_3 = \frac{13}{14}$.

(c) Again we use first step analysis (specifically Theorem 3.2 from the notes with $w(k) = k$). Let $W = \sum_{t=0}^{T-1} X_t$ and define

$$u_i = \mathbb{E}(W \mid X_0 = i).$$

We want to find u_3. Conditioning on the first step

$$\begin{aligned}
u_2 &= 2 + \tfrac{1}{2}u_3 \\
u_3 &= 3 + \tfrac{1}{6}u_2 + \tfrac{1}{6}u_3 + \tfrac{1}{6}u_4 \\
u_4 &= 4 + \tfrac{1}{2}u_3 + \tfrac{1}{2}u_4
\end{aligned}$$

From the last equation

$$u_4 = 8 + u_3$$

Substituting this in the second equation

$$u_3 = 3 + \tfrac{1}{6}u_2 + \tfrac{1}{6}u_3 + \tfrac{4}{3} + \tfrac{1}{6}u_3$$

which gives

$$\tfrac{2}{3}u_3 = \tfrac{13}{3} + \tfrac{1}{6}u_2$$

and so

$$u_2 = 4u_3 - 26.$$

Now using the first equation

$$4u_3 - 26 = 2 + \tfrac{1}{2}u_3$$

which gives $u_3 = \frac{56}{7} = 8$.

(d) Any vector of the form

$$\begin{pmatrix} p & 0 & 0 & 0 & 1-p \end{pmatrix}$$

with $0 \leqslant p \leqslant 1$ is an equilibirum distribution. This gives infinitely many equilibrium distributions.

(e) If we leave state 3 by going to state 5 then we can never return so

$$f_3 \leqslant 1 - p_{3,5} = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

If the first step from state 3 is to loop to state 3 then we are guaranteed to return to state 3 so

$$f_3 \geqslant p_{3,3} = \tfrac{1}{6}.$$

(f) Because 3 is a transient state, we know that $M_3 \sim \text{Geom}(1-f_3)$. So $\mathbb{E}(M_3) = \frac{1}{1-f_3}$. Using the bounds from part (e)

$$\mathbb{E}(M_3) \leqslant \frac{1}{1-\frac{1}{2}} = 2$$
$$\mathbb{E}(M_3) \geqslant \frac{1}{1-\frac{1}{6}} = \tfrac{6}{5}$$

2.

(a) (i) The number of arrivals in the interval $[7,5]$ (or equivalently $X(7)-X(5)$).

(ii) The number of arrivals in the interval $[3,5]$ (or equivalently $X(5)-X(3)$).

(iii) The number of arrivals in the interval $[0,1]$ conditional on there being 10 arrivals in the interval $[0,3]$.

(b) The random variable U takes values $0,1,2$.

We have $U=0$ if and only if there are no arrivals in $[0,3]$. So $\mathbb{P}(U=0)=e^{-6}$.

We have $U=1$ if and only if there is either at least 1 arrival in $[0,1]$ and no arrivals in $[1,3]$ or there is at least 1 arrival in $[2,3]$ and no arrivals in $[0,2]$. So

$$\mathbb{P}(U=1) = (1-e^{-2})(e^{-4}) + (1-e^{-2})(e^{-4}) = 2e^{-4} - 2e^{-6}$$

Otherwise, we have $U=2$ so the pmf is:

x	0	1	2
$\mathbb{P}(U=x)$	e^{-6}	$2e^{-4}-2e^{-6}$	$1-2e^{-4}+e^{-6}$

(c) (i) For $Y(t)=0$ we need that there has been either 0 or 1 arrival from the X process in $[0,t]$. So

$$\mathbb{P}(Y(t)=0) = e^{-2t} + 2te^{-2t} = (1+2t)e^{-2t}$$

(ii) If the first arrival happens at time T_1 then the cdf of T_1 is

$$F_{T_1}(t) = \mathbb{P}(T_1 \leqslant t) = 1 - \mathbb{P}(Y(t)=0) = 1-(1+2t)e^{-2t}$$

(iii) For a Poisson process we know that the first arrival time is an Exponential random variable. However this is not the cdf of an Exponential random variable so $Y(t)$ is not a Poisson process.

(iv) In the Thinning Lemma we require that events are deleted randomly with each arrival being deleted independently of what happens to other arrivals. This does not apply for $Y(t)$.

3.

(a) We need to solve

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix}$$

That is

$$\begin{aligned} w_1 &= \tfrac{1}{2}w_5 + \tfrac{1}{2}w_6 \\ w_2 &= \tfrac{1}{2}w_5 + \tfrac{1}{2}w_6 \\ w_3 &= \tfrac{1}{3}w_1 + \tfrac{1}{3}w_2 \\ w_4 &= \tfrac{2}{3}w_1 + \tfrac{2}{3}w_2 \\ w_5 &= \tfrac{3}{4}w_3 + \tfrac{1}{4}w_4 \\ w_6 &= \tfrac{1}{4}w_3 + \tfrac{3}{4}w_4 \end{aligned}$$

From the first two equations $w_1 = w_2$.

Now from the third equation $w_3 = \frac{2}{3}w_1$ and from the fourth equation $w_4 = \frac{4}{3}w_1$.

Finally

$$\begin{aligned} w_5 &= \tfrac{3}{4}w_3 + \tfrac{1}{4}w_4 = \tfrac{3}{4}\tfrac{2}{3}w_1 + \tfrac{1}{4}\tfrac{4}{3}w_1 = \tfrac{5}{6}w_1 \\ w_6 &= \tfrac{1}{4}w_3 + \tfrac{1}{4}w_4 = \tfrac{1}{4}\tfrac{2}{3}w_1 + \tfrac{3}{4}\tfrac{4}{3}w_1 = \tfrac{7}{6}w_1 \end{aligned}$$

So

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix} = \begin{pmatrix} w_1 & w_1 & \frac{2}{3}w_1 & \frac{4}{3}w_1 & \frac{5}{6}w_1 & \frac{7}{6}w_1 \end{pmatrix}$$

We also need $w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = 1$ and so take $w_1 = \frac{1}{6}$ giving unique equlibrium distribiution

$$\mathbf{w} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{9} & \frac{2}{9} & \frac{5}{36} & \frac{7}{36} \end{pmatrix}$$

(b) This chain does not have a limiting distribution. For there to be a limiting distribution we must have that P^r tends to a limit in which every row is the vector $\mathbf{w}$. But for any state s we have $p_{s,s}^{(r)} = 0$ when r is not a multiple of 3 so this cannot happen.

(c) By Theorem 4.9 in the notes, the expectation of the proportion of time the chain spends in state 1 up to time t tends to w_1 as $t \to \infty$. In this case it tends to $\frac{1}{6}$.

(d) Clearly if t is not a multiple of 3 then $\mathbb{P}(X_t = 1) = 0$.

If t is a multiple of 3 then we must have X_{t-1} is 5 or 6 and:

$$\mathbb{P}(X_t = 1) = \mathbb{P}(X_t = 1 \mid X_{t-1} = 5)p_{51} + \mathbb{P}(X_t = 1 \mid X_{t-1} = 6)p_{61}$$

But $p_{51} = p_{61} = \frac{1}{2}$ so

$$\mathbb{P}(X_t = 1) = \tfrac{1}{2}\left(\mathbb{P}(X_t = 1 \mid X_{t-1} = 5) + \mathbb{P}(X_t = 1 \mid X_{t-1} = 6)\right) = \tfrac{1}{2}$$

So

$$\mathbb{P}(X_t = 1) = \begin{cases} 0 & \text{if } t \neq 3k; \\ \frac{1}{2} & \text{if } t = 3k. \end{cases}$$

(e) $Q = \frac{1}{3}P + \frac{2}{3}I$ (where I is the 6×6 identity matrix). So

$$Q = \begin{pmatrix} 2/3 & 0 & 1/9 & 2/9 & 0 & 0 \\ 0 & 2/3 & 1/9 & 2/9 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 1/4 & 1/12 \\ 0 & 0 & 0 & 2/3 & 1/12 & 1/4 \\ 1/6 & 1/6 & 0 & 0 & 2/3 & 0 \\ 1/6 & 1/6 & 0 & 0 & 0 & 2/3 \end{pmatrix}$$

(f) This chain is regular because it is irreducible and contains a state s with $p_{s,s} > 0$. Since it is regular it has a limiting distribution by Theorem 4.7 from the notes.

(g) The way we have constructed Q means that the limiting distribution for the process Y_t is the same as the unique equilibrium distribution for X_t. So the probability that the chain is in an odd numbered state tends to $w_1 + w_3 + w_5 = \frac{1}{6} + \frac{1}{9} + \frac{5}{36} = \frac{5}{12}$ as $t \to \infty$.

4.

(a) (i) Let X_t be the number the counter is on after t rolls of the die. Then $(X_0, X_1, \dots)$ is a Markov chain with state space $\{1, 2, \dots, 7\}$ and transition matrix:

$$\begin{pmatrix} 1/2 & 1/6 & 0 & 0 & 0 & 0 & 1/3 \\ 1/3 & 1/2 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/2 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/2 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/2 & 1/6 \\ 1/6 & 0 & 0 & 0 & 0 & 1/3 & 1/2 \end{pmatrix}$$

(ii) At each step, the next move depends only on the current position of the counter and the die roll. Hence the Markov property is satisfied.

(iii) If I use a different biased die for each roll then the process will still be a Markov chain but will not be homogeneous.

(b) (i) Take states to be

- State 1: machine running
- State 2: waiting for engineer
- State 3: machine being repaired
- State 4: machine permanently broken

If $X(t)$ is the state of the machine at time t then $(X(t) : \ t \geqslant 0)$ is a continuous-time Markov chain with generator matrix

$$G = \begin{pmatrix} -\alpha & \alpha & 0 & 0 \\ 0 & -\beta & \beta & 0 \\ p\gamma & 0 & -\gamma & (1-p)\gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(In the third row, we need $\frac{g_{1,3}}{g_{1,3}+g_{3,4}} = p$. This together with the row sum being 0 determines the entries.)

(ii) Let B be the time of the first breakage. We have that $B \sim \text{Exp}(\alpha)$. So $\mathbb{P}(B \geqslant t) = e^{-\alpha t}$. We want $\mathbb{P}(B \geqslant 1)$ which is $e^{-\alpha}$.

(iii) The machine can break and be repaired at any time so it is more natural to track its state continuously rather than at discrete time steps.

(c) Many possible answers.

Example 1: I like Theorem 7.1 from the notes. This shows that the infinitesimal description of the Poisson process is equivalent to the definition involving the Poisson distribution. I like that this is proved using differential equation which seemed a surprising technique to use. I also like the fact that it gives two equivalent definitions each of which can be useful in different situations.

Example 2: I like the Superposition Lemma. This describes how two Poisson processes can be combined to give a new Poisson process. I like the fact that this arises naturally in many examples.