Random Processes – January 2022 exam Solutions

You are strongly encouraged to attempt the paper and to check your solutions without looking at these solutions. However, if you get really stuck on a question then it may be useful to have these. For some questions there may be several equally valid answers or aproaches so your answer might be correct even if it doesn't look exactly the same as mine.

1.

(a) (i) This is just $p_{1,2} = \frac{1}{4}$

(ii) $\mathbb{P}(X_2 = 1 \mid X_0 = 1) = p_{1,3}p_{3,1} + p_{1,4}p_{4,1} = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$.
(Equivalently, it is the top left entry of the matrix P^2.)
So $\mathbb{P}(X_2 \neq 1 \mid X_0 = 1) = 1 - \frac{5}{16} = \frac{11}{16}$.

(b) It is possible to get between any two states in exactly 3 steps (by inspection or calculating P^3) so the chain is regular. By Theorem 4.7 from the notes, every regular chain has a limiting distribution.

(c) By Theorem 4.6, if there is a limiting distribution then it is the unique equilibrium distribution. So we can find it by solving $\mathbf{w}P = \mathbf{w}$

(d) By the previous part we need to solve

$$\begin{aligned} w_1 &= \tfrac{1}{4}w_3 + \tfrac{1}{2}w_4 \\ w_2 &= \tfrac{1}{4}w_1 + \tfrac{3}{4}w_3 + \tfrac{1}{4}w_4 \\ w_3 &= \tfrac{1}{4}w_1 + \tfrac{3}{4}w_2 + \tfrac{1}{4}w_4 \\ w_4 &= \tfrac{1}{2}w_1 + \tfrac{1}{4}w_2 \end{aligned}$$

Subtracting the third equation from the second equation we get

$$\tfrac{7}{4}w_2 = \tfrac{7}{4}w_3 \quad \text{so } w_2 = w_3$$

Now subtracting the fourth equation from the first we get

$$\tfrac{3}{2}w_1 = \tfrac{3}{2}w_4 \quad \text{so } w_1 = w_4$$

Now from the first equation $w_3 = 2w_1$. So

$$\mathbf{w} = \begin{pmatrix} w_1 & 2w_1 & 2w_1 & w_1 \end{pmatrix}$$

But $\mathbf{w}$ is a probability vector so $w_1 + 2w_1 + 2w_1 + w_1 = 1$ so $w_1 = \frac{1}{6}$. So

$$\mathbf{w} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

(e) (i) Yes. This is $w_1 + w_3 = \frac{1}{2}$.

(ii) Yes. This is $w_4 = \frac{1}{6}$.

(iii) Yes. The expectation of the (first) return time to state 1 is $\frac{1}{w_1} = 6$. So by linearity of expectation the expectation of the time of the third visit (that is the second return) to state 1 is 12.

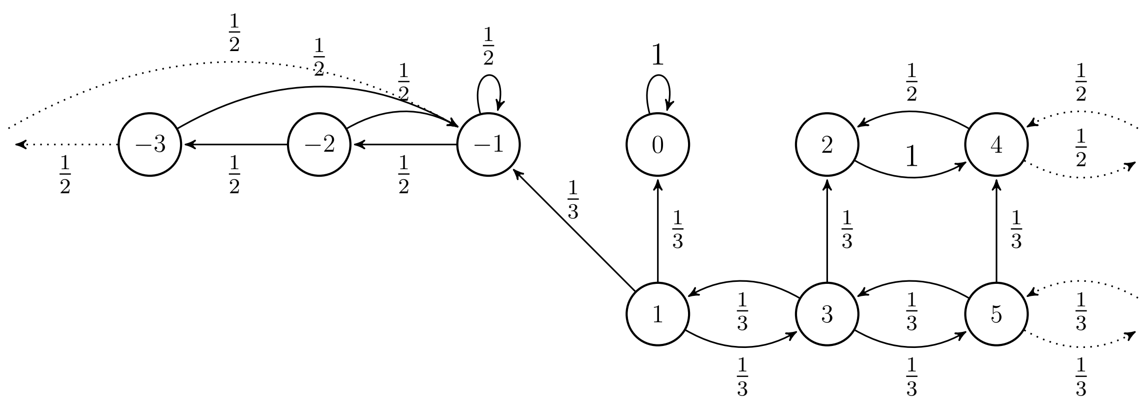
(iv) No. We could find this by first-step analysis. Change states 3 and 4 into absorbing states and use first-step analysis to find the probability that we are absorbed at 4. This is equal to the required probability.

2.

(a) Lots of possibilities. For instance:

- Every irreducible chain is recurrent.
- Every recurrent state is positive recurrent.
- Every recurrent chain has an equilibrium distribution.

(b) The transition graph is:



(c) Let O be the set of positive odd numbers. The chain starts at 1 which is in O. It will wander around O for some time but will eventually leave and can never return. When it leaves O this will be to one of

- state 0 in which case it is absorbed
- state -1 in which case it follows a copy of the success-run chain on $\{-1, -2, \ldots\}$

- a positive even numbered state in which case it follows a copy of the random walk on $\{2, 4, 6, \ldots\}$ with reflecting boundary at 2.

(d) The communicating classes are:

$$\{-1, -2, -3, \ldots\}, \quad \{0\}, \quad \{1, 3, 5, \ldots\}, \quad \{2, 4, 6 \ldots\}.$$

(e)
- State 0 is positive recurrent as it is absorbing
- State 1 is transient as $f_1 \leqslant 1 - \frac{1}{3} = \frac{2}{3} < 1$. Transience is a class property so every state in this $\{1, 3, 5, \ldots\}$ is transient.
- Every state in $\{2, 4, 6, \ldots\}$ is null recurrent because (from Examples 24 and 31 in lectures) the random walk on $\mathbb{N}$ with reflecting boundary is null recurrent.
- Every state in $\{-1, -2, -3, \ldots\}$ is positive recurrent because (from Examples 23 and 30 in lectures) the success runs chain is positive recurrent.

3. Let $S(i)$ count the number of solo bikes, $T(i)$ count the number of tandems, $B(i)$ count the total number of bikes, and $R(i)$ count the number of riders, I see up to time i.

(a) We have $B(i) = S(i) + T(i)$ so we are superimposing (summing) two Poisson processes. By a Theorem from lectures the result is a Poisson process. Specifically $B(i)$ is a Poisson process of rate 6.

(b) The interarrival times for $B(i)$ are distributed Exp(6) so each has expectation $\frac{1}{6}$. So the expectation of time the third bicycle passes (which is the sum of the first three interarrival times) is $\frac{3}{6} = \frac{1}{2}$

(c) (i) Since $T(t)$ is a Poisson process of rate 1, $T(t) \sim \text{Po}(t)$ and

$$\mathbb{P}(T(t) = 2) = e^{-t}\frac{t^2}{2!} = \frac{t^2}{2e^t}$$

(ii) Since $B(t)$ is a Poisson process of rate $6t$, $B(t) \sim \text{Po}(6t)$ and

$$\mathbb{P}(B(t) = 2) = e^{-6t}\frac{(6t)^2}{2!} = \frac{18t^2}{e^{6t}}$$

(iii) This can happen in two ways; 2 solo bikes, or 1 tandem. So

$$\begin{aligned}\mathbb{P}(R(t)=2) &= \mathbb{P}(S(t)=2, T(t)=0) + \mathbb{P}(S(t)=0, T(t)=1) \\ &= e^{-5t}\frac{(5t)^2}{2!}e^{-t}\frac{t^0}{0!} + e^{-5t}\frac{(5t)^0}{0!}e^{-t}\frac{t^1}{1!} \\ &\qquad \text{(since } S \text{ and } T \text{ are independent processes)} \\ &= e^{-6t}\frac{25t^2}{2} + e^{-6t}t \\ &= \left(\frac{25t^2}{2} + t\right)e^{-6t}\end{aligned}$$

(d) R is not a Poisson process. Either because the probability in the previous part has the wrong form. Or the probability that $R(i)$ increases by 2 in an interval of length h is not $o(h)$ (because there is probability $h + o(h)$ that a tandem passes in this interval.

(e) No. If $R(i)$ were a birth process we would have $\mathbb{P}(R(i+h)=2 \mid R(i)=0) = o(h)$ but (as above) this is not true.

(f) Chapman-Kolmogorov says

$$\begin{aligned}p_{0,j}(t+h) &= \sum_{k\in\mathbb{N}} p_{0,k}(t)p_{k,j}(h) \\ &= p_{0,j}(t)p_{j,j}(h) + p_{0,j-1}(t)p_{j-1,j}(h) + p_{0,j-2}(t)p_{j-2,j}(h) + o(h) \\ &\qquad \text{(for all } k \text{ except } j-2, j-1, j \text{ the summand is } o(h)) \\ &= p_{0,j}(t)(1-6h+o(h)) + p_{0,j-1}(t)(5h+o(h)) + p_{0,j-2}(t)(h+o(h)) + o(h) \\ &\text{(the only significant contribution to } p_{j-2,j}(h) \text{ is from the event that 1 tandem passes)} \\ &= p_{0,j}(t) - 6hp_{0,j}(t) + 5hp_{0,j-1}(t) + hp_{0,j-2}(t) + o(h)\end{aligned}$$

So

$$\begin{aligned}\frac{1}{h}\left(p_{0,j}(t+h) - p_{0,j}(t)\right) &= \frac{1}{h}\left(-6hp_{0,j}(t) + 5hp_{0,j-1}(t) + hp_{0,j-2}(t)\right) \\ &= -6p_{0,j}(t) + 5p_{0,j-1}(t) + p_{0,j-2}(t) + o(1)\end{aligned}$$

Letting $h \to 0$ we get

$$p'_{0,j}(t) = -6p_{0,j}(t) + 5p_{0,j-1}(t) + p_{0,j-2}(t)$$

4.

(a) We would let X_k be the number of points accumulated by the end of match k. The state space will be $\mathbb{N}$. The team starts with 0 points so we take $X_0 = 0$. If $X_k = i$ then we have

- If match $k+1$ is won then $X_{k+1} = i+3$ this happens with probability p_w.
- If match $k+1$ is lost then $X_{k+1} = i$ this happens with probability p_l.
- If match $k+1$ is drawn then $X_{k+1} = i+1$ this happens with probability $1 - p_w - p_l$.

So

$$p_{i,j} = \begin{cases} p_w & \text{if } j = i+3 \\ 1 - p_w - p_l & \text{if } j = i+1 \\ p_l & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

(b) If we do this the process is still a Markov chain but it is no longer homogeneous. We have

$$\mathbb{P}(X_1 = 3 \mid X_0 = 0) = \mathbb{P}(\text{the first match is won})$$
$$\mathbb{P}(X_2 = 3 \mid X_1 = 0) = \mathbb{P}(\text{the second match is won})$$

and these are not necessarily equal.

(c) If we do this the process is no longer a Markov chain. For instance

$$\mathbb{P}(X_3 = 6 \mid X_2 = 3, X_1 = 3, X_0 = 0) \neq \mathbb{P}(X_3 = 6 \mid X_2 = 3, X_1 = 0, X_0 = 0)$$

since the first of these requires the team to win the third match having lost the second, while the second requires the team to win the third match having won the second. We are told that these are different.

(d) (i) Let $F(i)$ be the number of flaws in the first i metres of cable. It is natural to model $F(i)$ as a Poisson process of rate α per metre.

(ii) The interarrival times are the distances between consecutive flaws.

(iii) The undetected flaws will form a Poisson process of rate $\alpha(1-\beta)$ per metre. This is an example of thinning a Poisson process.