Example 7. Consider the system

$$\dot{x} = -x + y^2,$$
$$\dot{y} = -2y + 3x^2.$$

at origin $\dot{x} = -x$, $\dot{y} = -2y$

This system has two equilibria, one at the origin, which is locally asymptotically stable (by linearization), and hence another one unstable (why?). Can we somehow estimate the basin of attraction of $(0,0)$? Consider the Lyapunov function

stable node

$$V(x,y) = \frac{x^2}{2} + \frac{y^2}{4},$$

$\lambda_1 = -1$

$\lambda_2 = -2$

which is positive definite. Now calculate

$$\dot{V}(x,y) = -x^2 + xy^2 - y^2 + \frac{3}{2}yx^2 = -x^2(1 - \frac{3}{2}y) - y^2(1-x),$$

$\frac{\lambda_2}{\lambda_1} > 1$

which is negative for all (x,y) that satisfy $x < 1$ and $y < 2/3$. This clearly indicates, as we know, that the origin is asymptotically stable. To gain an idea of the basin of attraction, we must find the largest region around $(0,0)$ where $V(x,y) \leq \alpha$ and still be negative definite. Since the level sets of V are the ellipses with the axes $\sqrt{2\alpha}$ and $2\sqrt{\alpha}$ hence we must have that $\sqrt{2\alpha} < 1$ and $2\sqrt{\alpha} < 2/3$, which implies that $\alpha < 1/9$, which means that the largest region that we can be sure lays inside the basin of attraction is the ellipse

$$\frac{x^2}{2} + \frac{y^2}{4} = \frac{1}{9}.$$

However, as numerical illustration shows, the actual basin of attraction is way bigger, but still smaller than the whole plane (see Fig. 2).

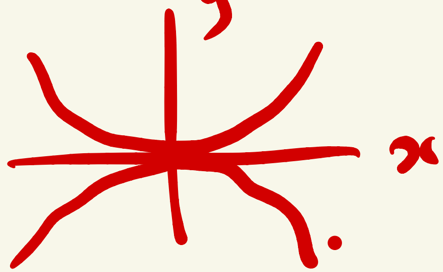
$\dot{x} = \lambda_1 x$

$\dot{y} = \lambda_2 y$

$\frac{dy}{y} = \frac{\lambda_2}{\lambda_1}\frac{dx}{x} \Rightarrow \ln y = \frac{\lambda_2}{\lambda_1}\ln x + c$

$y = C' x^{\lambda_2/\lambda_1}$

λ_2/λ_1



$V(x,y) = x^2 + y^2$ PD

$\dot{V}(x,y) = -x^2 + xy^2 - y^2 + \frac{3}{2}x^2 y$

$= -x^2\left(1 - \frac{3y}{2}\right) - y^2(1-x)$

$4 -$

$1 - \frac{3y}{2} > 0$

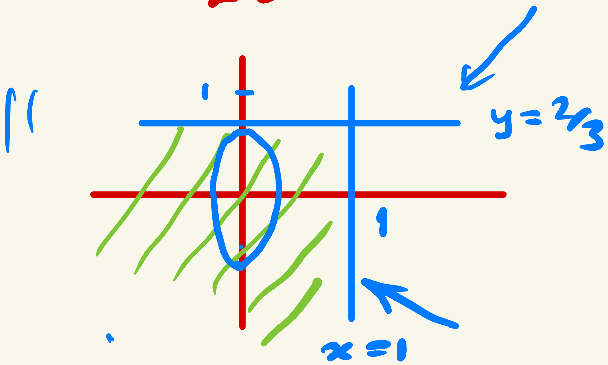
$1 - x > 0$

$\dot{x} = -x + y^2, \quad \dot{y} = -2y + 3y^2$

Try $V(x,y) = \frac{x^2}{2} + \frac{y^2}{4}$

$\dot{V}(x,y) = -x^2\left(1 - \frac{3y}{2}\right) - y^2(1-x)$

$x < 1, \quad \frac{3}{2}y < 1$



$\dot{V}$ is ND in the "green" region

But $L = $ constant $(= \alpha)$ curves ellipses. x semi-axis $\sqrt{2\alpha}$

y semi-axis $2\sqrt{\alpha}$ larger.

max α for L?

$$\boxed{V = \frac{x^2}{2} + \frac{y^2}{4} = 1}$$

$V = $ constant

$V = \alpha, \quad \frac{x^2}{2} < \alpha, \quad \frac{y^2}{4} < \alpha$

$x < \sqrt{2}\sqrt{\alpha} \quad \underline{y < 2\sqrt{\alpha}.}$

$x < 1$

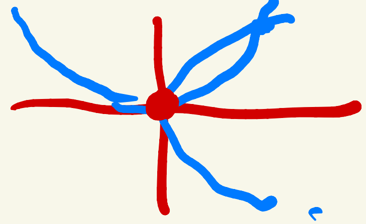
$y < \frac{2}{3}$

$\sqrt{2}\sqrt{\alpha} < 1 \qquad \sqrt{\alpha} < \frac{1}{\sqrt{2}} \qquad \alpha < \frac{1}{2}$

$\cancel{2}\sqrt{\alpha} < \frac{\cancel{2}}{3} \qquad \sqrt{\alpha} < \frac{1}{3} \qquad \alpha < \frac{1}{9}$

$\frac{x^2}{2} + \frac{y^2}{4} < \frac{1}{9}$

Ex $\dot{x} = -x + y^2$ $\quad \dot{y} = -2y + 3x^2$ $\quad \dot{x} = 0,\ \dot{y} = 0$



Locally asymptotically stable at the origin

Lin. $\dot{x} = -x \quad , \dot{y} = -2y$ (stable node!)

Fixed points : $\dot{x} = 0 \ , \dot{y} = 0$

$\dot{x} = 0 \Rightarrow x = y^2 , \Rightarrow \dot{y} = -2y + 3y^4 = 0$

$* \quad y(3y^3 - 2) = 0^{*} , \quad y = \sqrt[3]{\frac{2}{3}} \ , \ x = \sqrt[3]{\frac{4}{9}}$ Saddle

$J = \begin{bmatrix} -1 & 2y \\ 6x & -2 \end{bmatrix}$ ✓ $\therefore J = \begin{bmatrix} -1 & 2\sqrt[3]{\frac{2}{3}} \\ 6\sqrt[3]{\frac{4}{9}} & -2 \end{bmatrix}$ at $(x,y) = \left(\sqrt[3]{\frac{4}{9}} , \sqrt[3]{\frac{2}{3}}\right)$ 0.76

0.76 $\quad$ 0.87

12xy

$6\sqrt[3]{\frac{4}{9}} \cdot 2\sqrt[3]{\frac{2}{3}} = 12 \cdot \sqrt[3]{\frac{8}{27}}$

$= 12 \cdot \frac{2}{3} = 8$

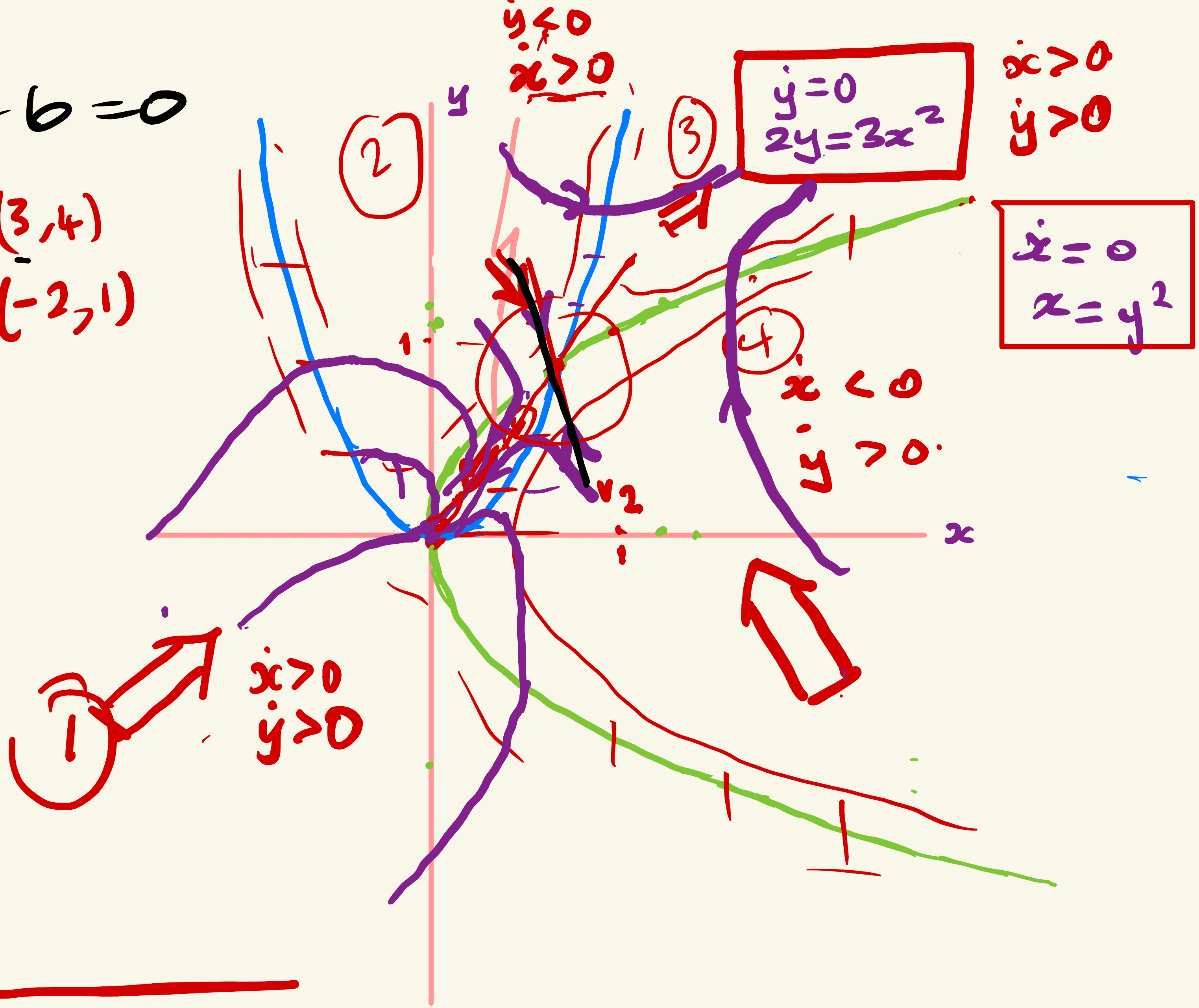
$|J| = -6$

$\therefore$ saddle.

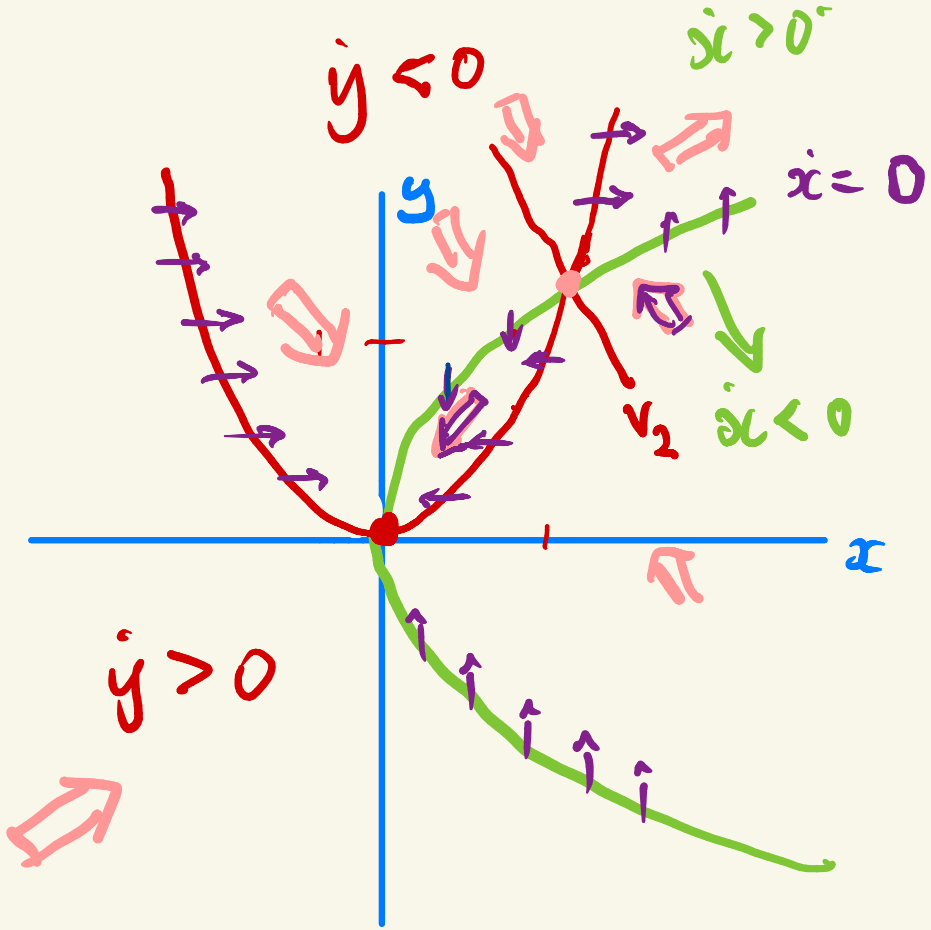
E-eq^n: $\lambda^2 + 3\lambda - 6 = 0$

$\lambda_1 \approx 1.4 \quad \underline{v}_1 \approx (3,4)$

$\lambda_2 = -4.4 \quad \underline{v}_2 \approx (-2,1)$



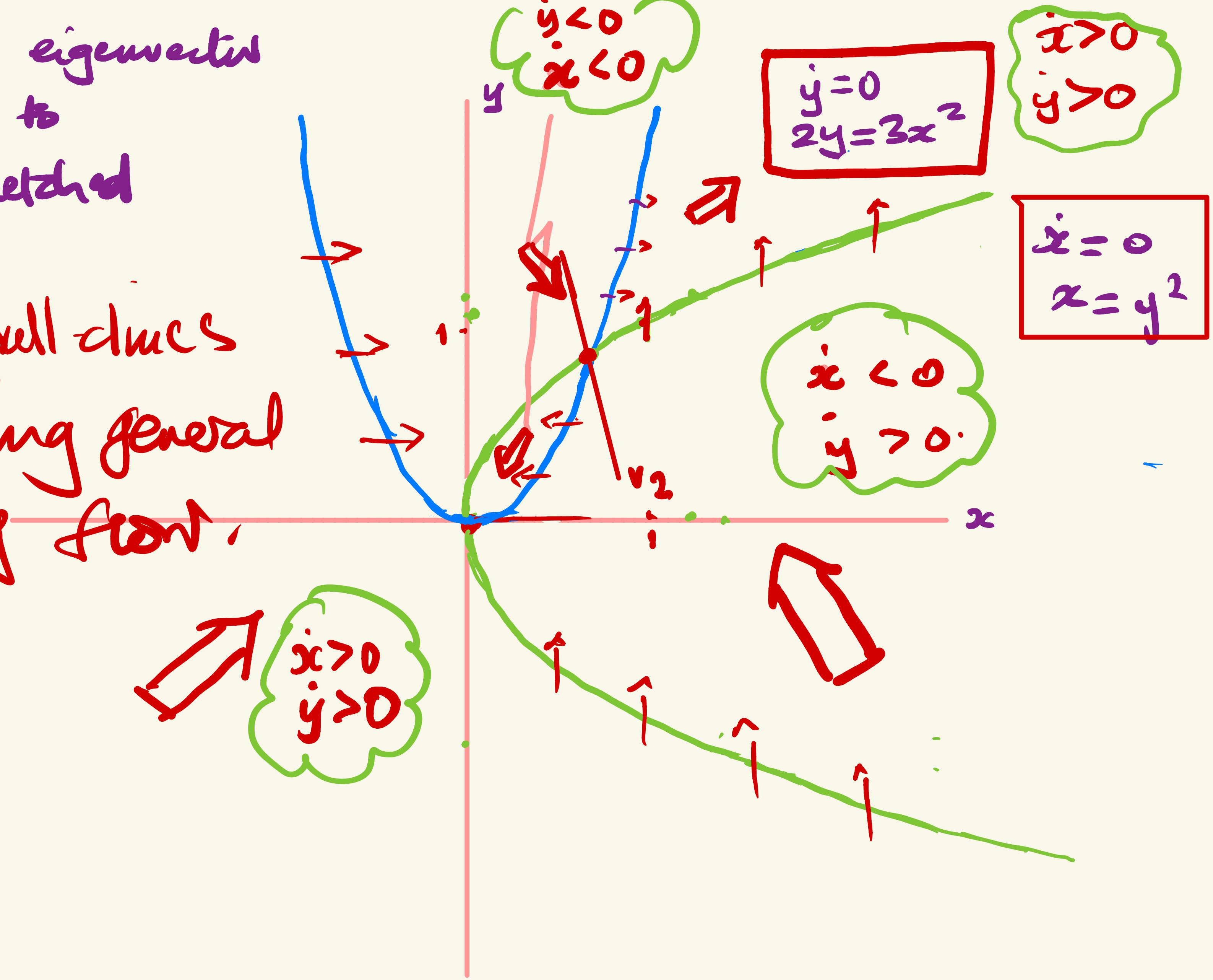
$$\lambda = \frac{-3 \pm \sqrt{9 + 4 \cdot 6}}{2} = \frac{-3 \pm \sqrt{33}}{2}$$



Completing null-clines and deciding general directions of flow

The stable v_2 eigenvector corresponding to $x_2 \approx -3$ is sketched

Completing null clines and deciding general directions of flow.

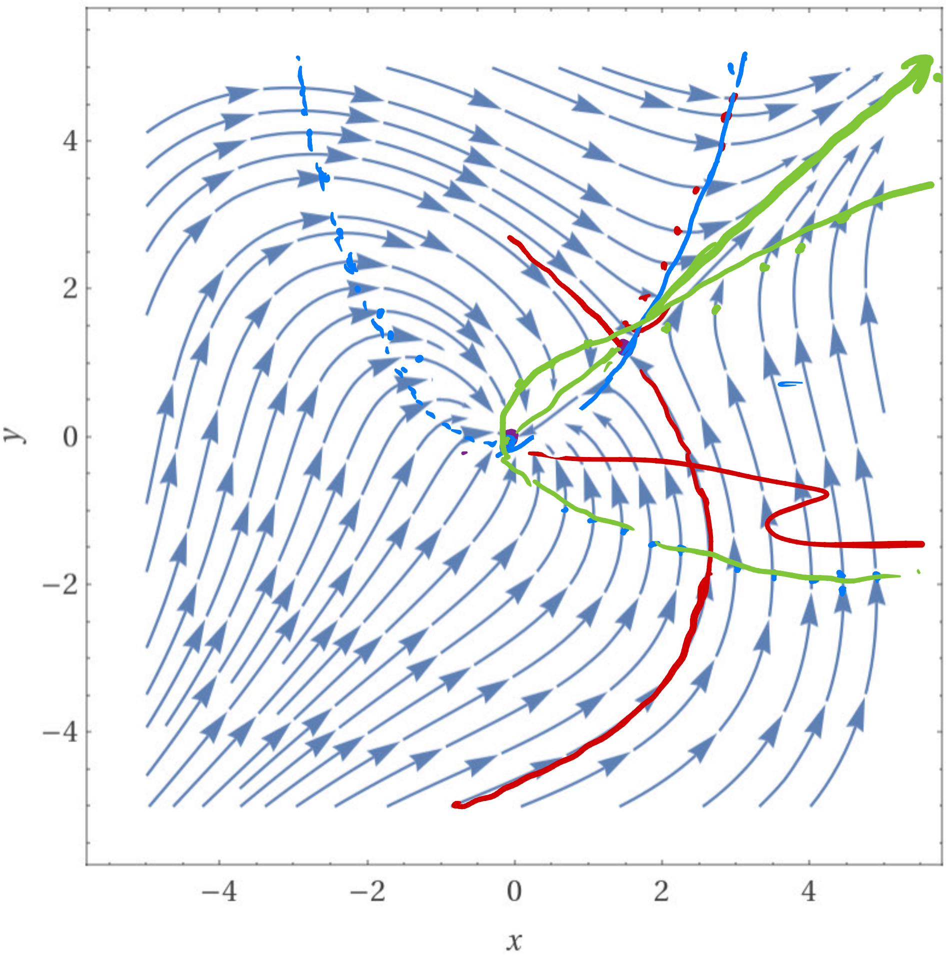


Streamplot{-x+y^2,-2y+x^2} x from-5 to +5, y from -5 to 5

Input interpretation

stream plot	$(-x+y^2, x^2-2y)$	$x = -5$ to 5
		$y = -5$ to 5

Plot



At
(0,0)
$\frac{\lambda_2}{\lambda_1} = \frac{2}{1}$

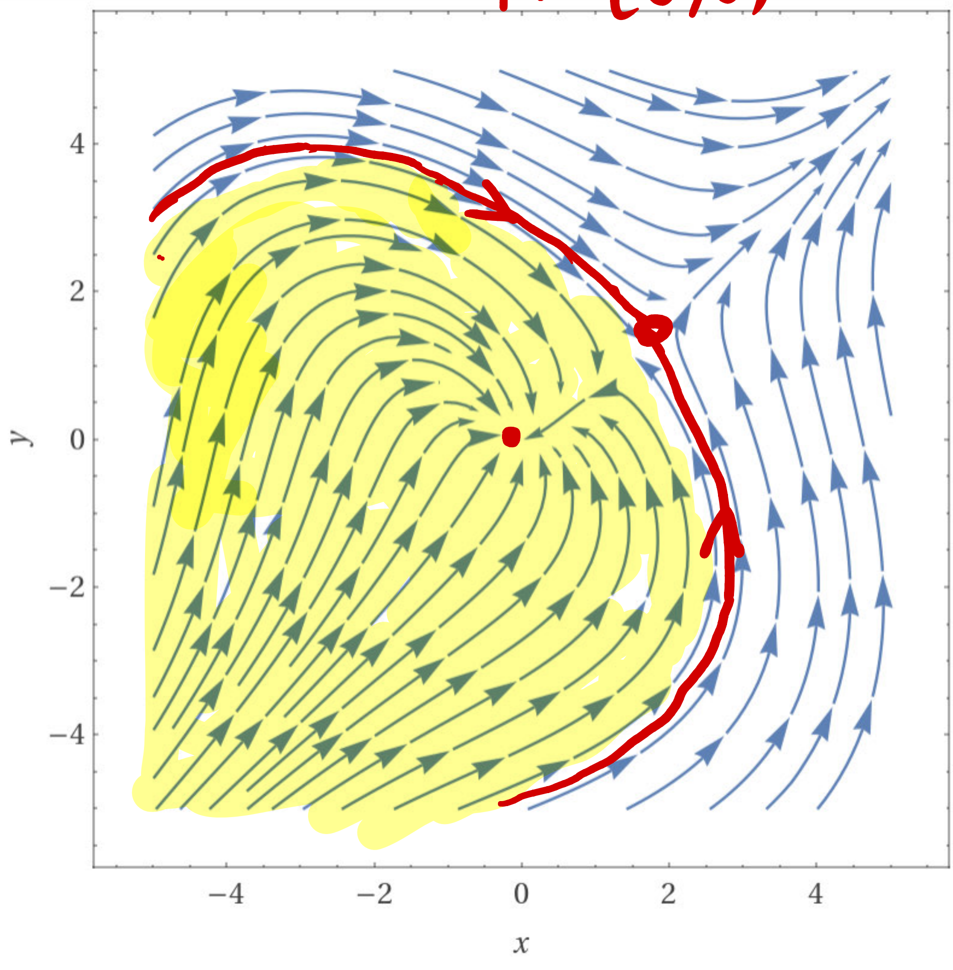
Streamplot{-x+y^2,-2y+x^2} x from-5 to +5, y from -5 to 5

Input interpretation

stream plot	$(-x + y^2, x^2 - 2y)$	$x = -5$ to 5 $y = -5$ to 5

Basin of attraction of fixed pt (0,0)

Plot



Ex $\dot{x} = y, \ \dot{y} = -\sin x$

$\ddot{x} = -\sin x$ NL pendulum.

Form of Newton II: $\ddot{x} = \dot{y} = -\sin x = -\dfrac{\partial V}{\partial x}$

$-\dfrac{\partial V}{\partial x} = -\sin x \Rightarrow V = C - \cos x$

$\frac{1}{2}y^2 + C - \cos x = E(x,y)$

(KE + PE = constant.)

$\dot{x} = 0, \ y = 0$ (vertical motion)

$\dot{y} = 0, \ -\sin x = 0 \Rightarrow x = n\pi$

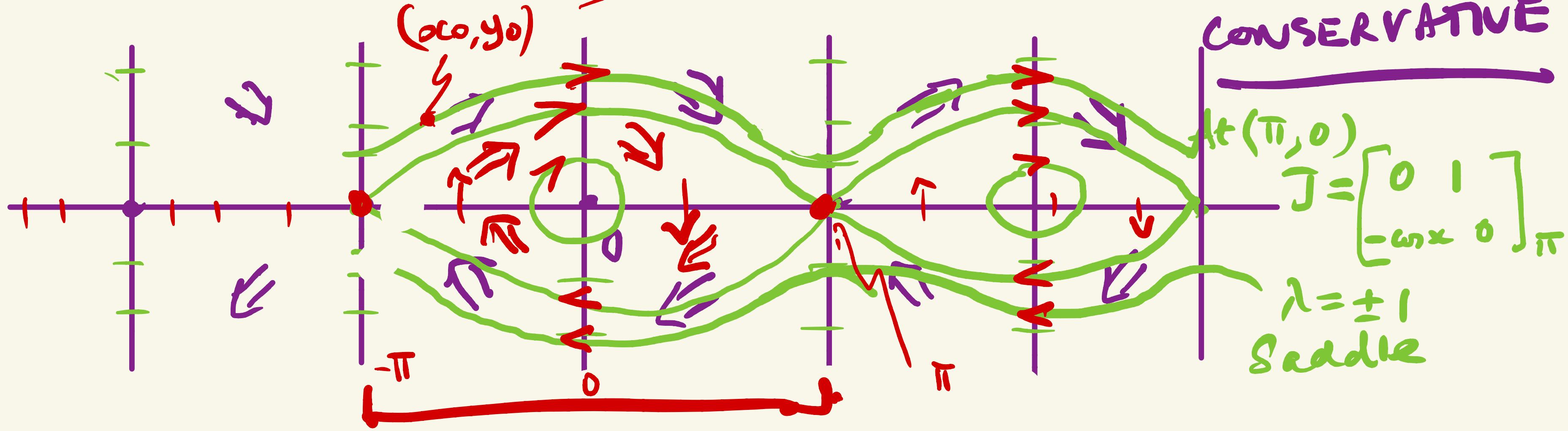
$E_0 = \frac{1}{2}y_0^2 - \cos x_0$

At $x = (0,0)$ Linearisation is a centre $\dot{x} = y, \ \dot{y} = -x$ ($\sin x \approx x$)

Same at all $(2n\pi, 0)$.

Also non-linear centre because CONSERVATIVE

$\Rightarrow$



$\dot{x} = \underline{y}$, $\dot{y} = -\sin x$

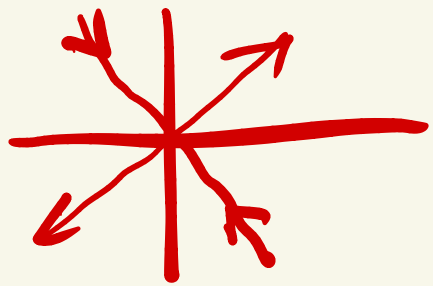
$y = 0$, $x = \pi$

$$J = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}\Bigg|_{x=\pi,\, y=0} = \begin{bmatrix} 0 & 1 \\ +1 & 0 \end{bmatrix}$$

Eigenvalues $\lambda^2 - 1 = 0$ $\lambda = 1, -1$

$\lambda = +1$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix}$ $y = x$ $(1, 1)$

$\lambda = -1$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$ $y = -x$ $(1, 1)$



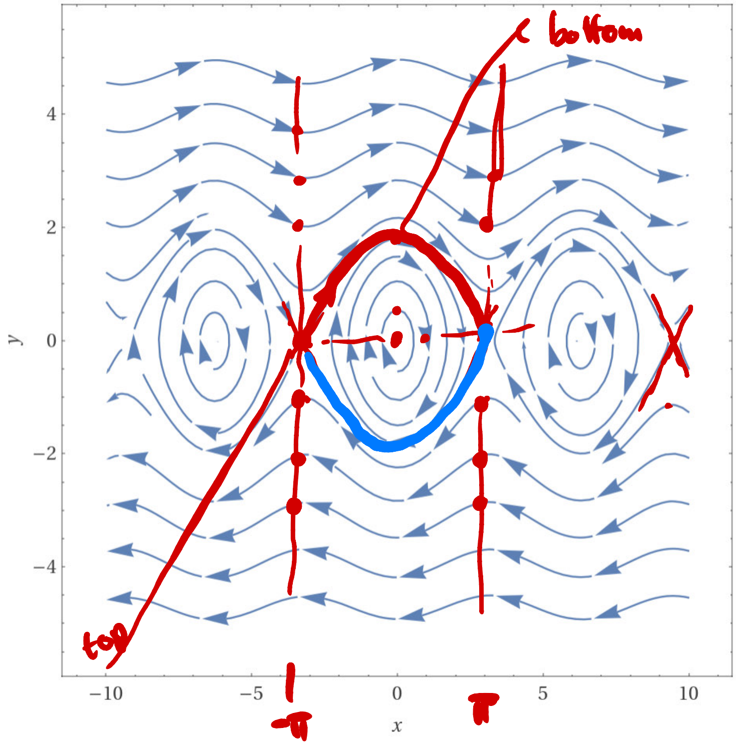
Conservation ensures saddles are connected.

streamplot{y,-sin(x)} x=—10 to x=10; y= -5 to y=5

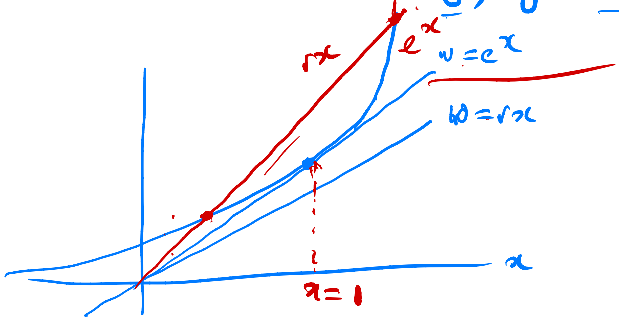
Input interpretation

stream plot	$(y, -\sin(x))$	$x = -10$ to 10
		$y = -5$ to 5

Plot



Ex $\ddot{x} = rx - e^x$ $\qquad \dot{x} = y, \ \dot{y} = rx - e^x.$



tangency of $w = rx$, $w = e^x$ is given by

$0 < r < e$ 0 int.

$r = e$. 1 intersection

$r > e$ 2 int.

$r > e$ $\qquad x_1^* < 1 < x_2^*$

$$\dot{y} = rx - e^x \begin{cases} < 0 & x_2^* < x \\ > 0 & x_1^* < x < x_2^* \\ < 0 & x < x_1^* \end{cases}$$

$e^x = rx$

$e^x = r$. $\quad x = \ln r$

$e^{\ln r} = r \ln r$.

$r = r \ln r$

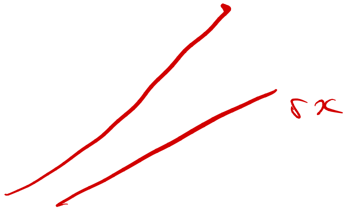
$r = 0$, $\ln r = 1$, $r = e$.

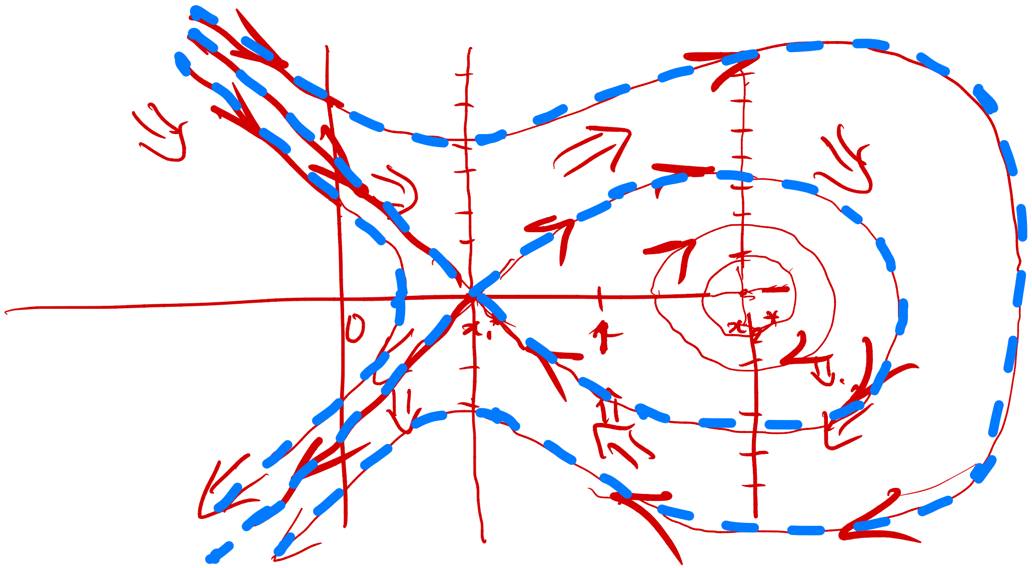
$\dot{x} = y$

$\dot{y} = rx - e^x$

$\dot{x} = y$

$\dot{y} = rx - e^x$

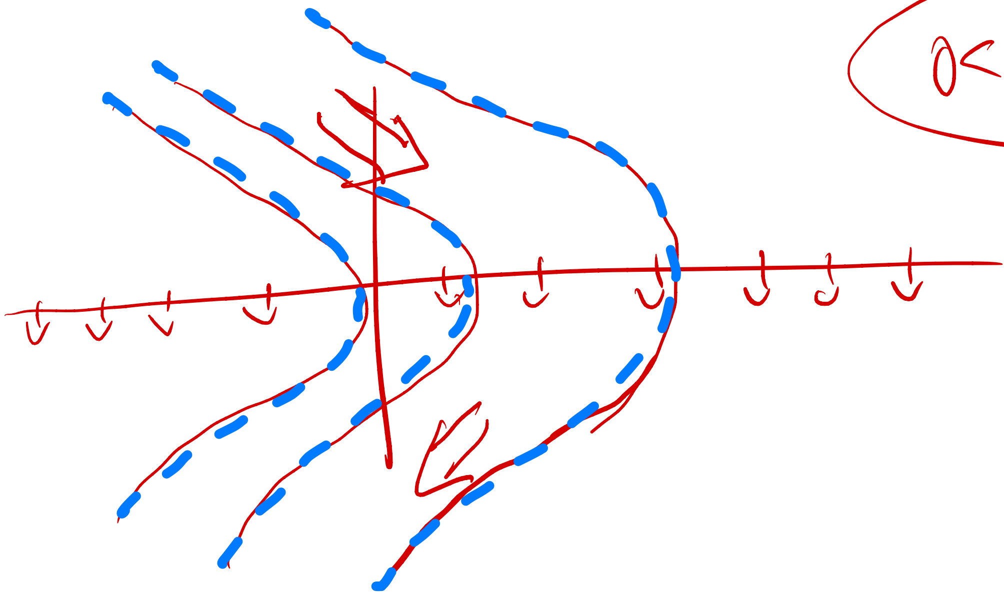




r>e

2 fixed pt

$-\frac{\partial V}{\partial x} = rx - e^x$



0<r<e

0 fixed pts

Example 8. Now let us consider again our familiar model of the pendulum

$$\ddot{x} + \sin x = 0,$$

which can be written as the system of two first order equations

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\sin x.\end{aligned}$$

We already know the phase portrait of this system (see Lecture 9), but here let me use the knew machinery of Lyapunov functions to establish that the origin is Lyapunov stable.

As a candidate of Lyapunov function let me take

$$V(x, y) = \frac{y^2}{2} + 1 - \cos x.$$

Note that in a small neighborhood of $(0, 0)$ my V is positive definite. Now

$$\dot{V}(x, y) = y \sin x + y(-\sin x) = 0,$$

and hence my V is an example of a Lyapunov function, but not strict Lyapunov function. Therefore, I can conclude, as I already know, that the origin is Lyapunov stable.

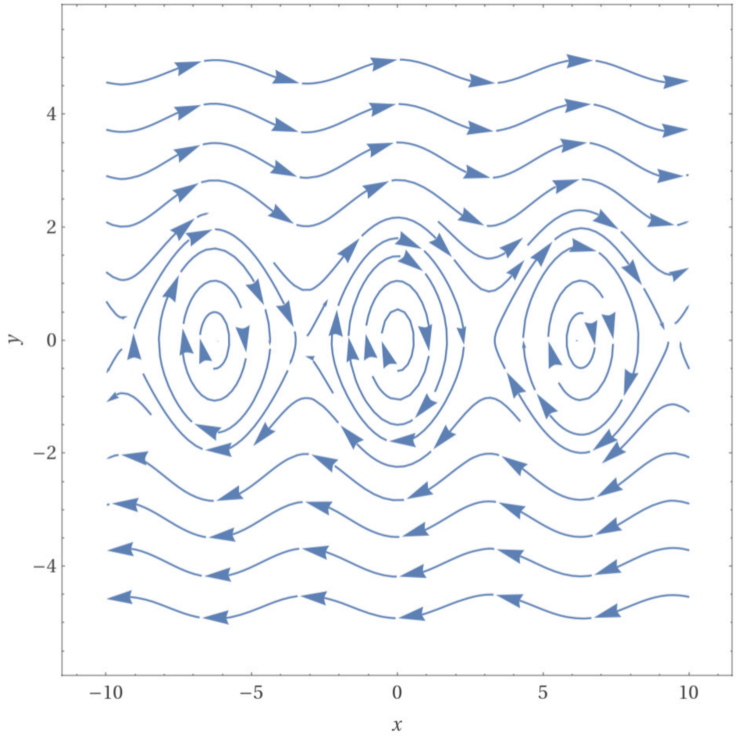
Wolfram

streamplot{y,-sin(x)} x=—10 to x=10; y= -5 to y=5

Input interpretation

stream plot	$(y, -\sin(x))$	$x = -10$ to 10
		$y = -5$ to 5

Plot



Recommend this app   Feedback

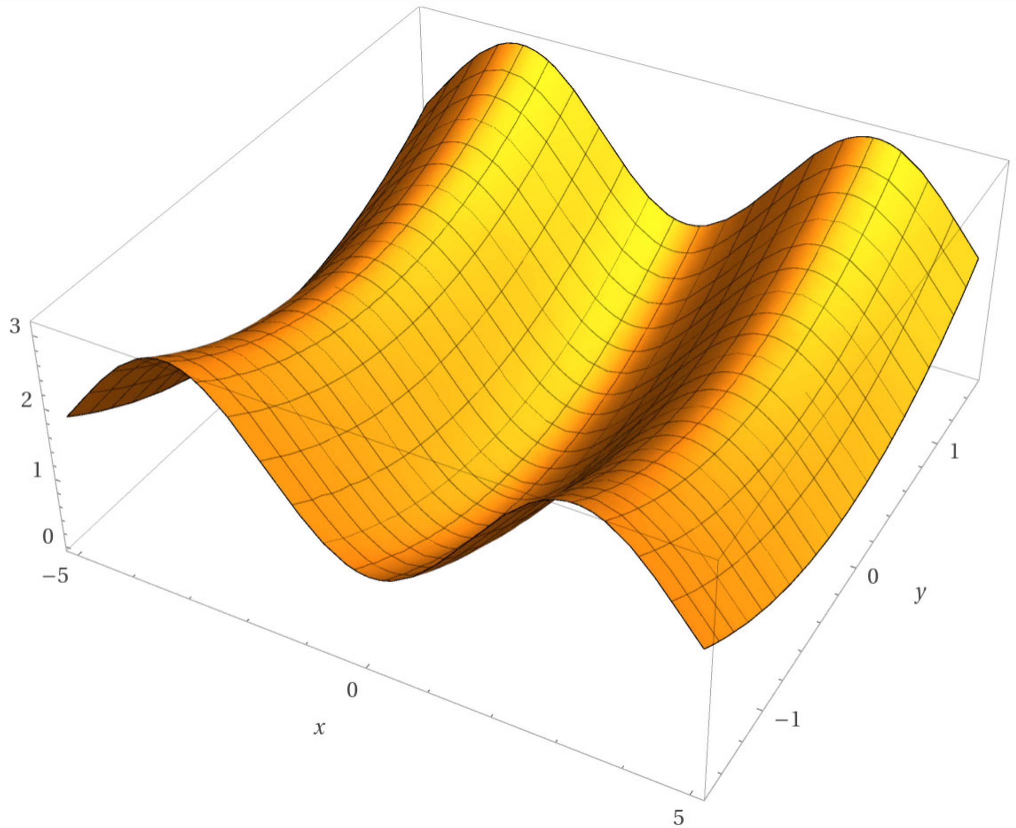
WolframAlpha

Plot (y^2)/2+ 1-cos(x) for x=-5 to x=5 and y=-1.5 to 1.5

Input interpretation

plot	$\frac{y^2}{2} + 1 - \cos(x)$	$x = -5$ to 5 $y = -1.5$ to 1.5

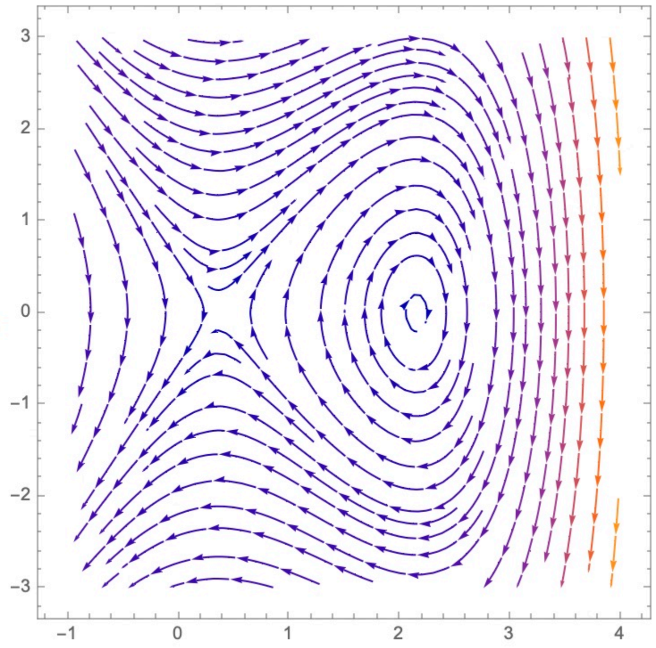
3D plot



Show contour lines

In[25]:= `r = 4; StreamPlot[{y, r*x - Exp[x]}, {x, -1, 4}, {y, -3, 3}]`

Out[25]=



In[20]:= `r = 2; StreamPlot[{y, r*x - Exp[x]}, {x, -1, 4}, {y, -3, 3}]`

Out[20]=