ASSESSED COURSEWORK 2

A1.

$$\begin{array}{ccc}
\alpha = \lfloor \sqrt{n(n+1)} \rfloor = n & \Rightarrow & \rho_1 = \dfrac{1}{\sqrt{n(n+1)} - n} = \dfrac{\sqrt{n(n+1)} + n}{n} \\
 & \swarrow & \\
\alpha_1 = \lfloor \dfrac{\sqrt{n(n+1)} + n}{n} \rfloor = 2 & \Rightarrow & \rho_2 = \dfrac{1}{\frac{\sqrt{n(n+1)}+n}{n} - 2} = \dfrac{1}{\frac{\sqrt{n(n+1)}-n}{n}} = \dfrac{n}{\sqrt{n(n+1)} - n} = \sqrt{n(n+1)} + n \\
 & \swarrow & \\
\alpha_2 = \lfloor \sqrt{n(n+1)} + n \rfloor = 2n & \Rightarrow & \rho_3 = \dfrac{1}{\sqrt{n(n+1)} + n - 2n} = \dfrac{1}{\sqrt{n(n+1)} - n} = \rho_1 \\
 & \swarrow & \\
\alpha_3 = \alpha_1 & \Rightarrow & \rho_4 = \rho_2 \\
 & \swarrow & \\
 & \vdots &
\end{array}$$

Hence $\sqrt{n^2+2} = [n, 2, 2n, 2, 2n, \ldots] = [n; \overline{2, 2n}]$.

A2.

Let $r = [1; 6, 1, 6, \ldots]$. By definition,

$$r = 1 + \frac{1}{6 + \frac{1}{r}},$$

hence $6r^2 - 6r - 1 = 0$. By the quadratic formula, we have $r = \dfrac{3+\sqrt{15}}{6}$. Hence

$$[4; \overline{1, 6}] = 4 + \frac{1}{[1; 6, 1, 6, \ldots]} = 4 + \frac{1}{\frac{3+\sqrt{15}}{6}} = 1 + \sqrt{15}.$$

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A3.

By Theorem 44, it suffices to establish that the inquality

$$|e - \frac{2721}{1001}| < \frac{1}{2(1001)^2}$$

holds. Since $e = 2.71828182845...$, observe

$$|e - 2.71828171828...| < 0.00000011017 < 0.0000004... < \frac{1}{2004002} = \frac{1}{2(1001)^2}.$$

A4. We firstly compute that the continued fraction for $\sqrt{29}$ is

$$[5; \overline{2, 1, 1, 2, 10}]$$

with $l = 5$. It follows from Theorem 48 that the fundamental solution for $x^2 - 29y^2 = \pm 1$ is given by $(s_{l-1}, t_{l-1}) = (s_4, t_4) = (70, 13)$ [we may either use the recursive definition of (s_n, t_n) to work out $(s_1, t_1) = (11, 2), (s_2, s_3) = (16, 3), (s_3, t_3) = (27, 5)$, or directly compute the 4-th convergent $r_4 = \frac{s_4}{t_4} = 5 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}$].

Since $s_4^2 - 29t_4^2 = 70^2 - 29 \cdot 13^2 = -1$, it is necessary to make appeal to Theorem 51 to find $(v_2, w_2) \in \mathbb{N} \times \mathbb{N}$ satisfying

$$v_2 + w_2\sqrt{29} = (s + t\sqrt{29})^2$$

where (s, t) is the fundamental solution $(s_4, t_4) = (70, 13)$, because the pair satisfies

$$v_2^2 - 29w_2^2 = (-1)^2 = 1;$$

in fact we know that $(v_2, w_2) = (s_{2 \cdot 5 - 1}, t_{2 \cdot 5 - 1}) = (s_9, t_9)$ and is the smallest solution to $x^2 - 29y^2 = 1$.

As $(70 + 13\sqrt{29})^2 = 9801 + 1820\sqrt{29}$, we know

$$(s_9, t_9) = (9801, 1820).$$

Of course it is perfectly fine to compute (s_9, t_9) by hand, but the point of this exercise to see that this 'technique' would allow us to compute convergent $r_n = \frac{s_n}{t_n}$ rather quickly even if n is large.