Comments concerning exam

Understand that:

1) Example $\mathbb{P}(Y^2 > X > 0) = \int_0^\infty \left(\int_{\sqrt{x}}^\infty f_{X,Y}(x,y)\,dy \right) dx$

$$= \int_0^\infty \left(\int_0^{y^2} f_{XY}(x,y)\,dx \right) dy$$

Know what is $f_{X|Y=y}$, $f_{Y|X=x}$.

2) Understand: $\mathbb{E}(g(X)|Y=y) = \int_{-\infty}^{\infty} g(x) f_{X|Y=y}(x)\,dx$

3) Be able to prove all statements we proved in lectures. Examples:

a) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

(b) X, Y are independent $\Leftrightarrow f_{X,Y}(x,y) = h(x)g(y)$.

(c) $X_1, \dots, X_n$ are normal r.v.'s, $X_i \sim N(\mu_i, \sigma_i^2)$.

Then $\sum_{i=1}^n X_i \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$.

(d) $X_i \sim Exp(\lambda)$, What is the pdf of $\sum_{i=1}^n X_i$

(e) Inequalities: Markov, Chebyshev.

(f) LLN. Etc.

In other words: Know all proofs!

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$M_{\underbrace{X_1+X_2+\dots+X_n}_{\text{independent}}}(t) = M_{X_1}(t)\, M_{X_2}(t) \dots M_{X_n}(t).$$

$$f_X(x) \longleftrightarrow M_X(t)$$

<u>Example</u> $X \sim Ga(\alpha, \beta) \iff f_X(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, x$

$$M_X(t) = \frac{\beta^\alpha}{(\beta - t)^\alpha}$$

$$\chi^2(1) \sim \xi^2, \quad \xi \sim N(0,1).$$

$$\chi^2(n) \sim Z = X_1^2 + X_2^2 \dots X_n^2, \quad X_i \sim N(0,1)$$

<u>Question</u>, (1) Find $M_{\chi^2(n)}(t) \equiv M_Z(t)$

(2) Find $f_{\chi_2(n)}(x) \equiv f_Z(x)$

$$M_Z(t) = M_{X_1^2}(t) \times M_{X_2^2}(t) \times \dots M_{X_n^2}(t) = (M_{X^2}(t))^n$$

$$X_i^2 \sim f_{X_i^2}(x) = \frac{1}{\sqrt{2}\,\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{-\frac{1}{2}x}$$

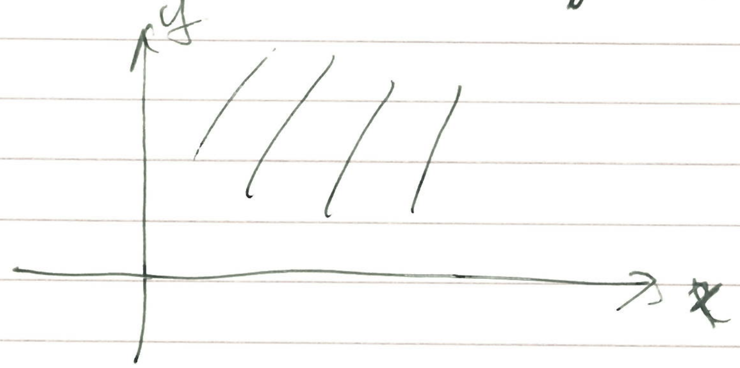
$$M_{X_i^2}(t) = \frac{1}{\sqrt{2}\,(\frac{1}{2} - t)^{\frac{1}{2}}} \quad \text{Hence}$$

$$M_Z(t) = \frac{1}{2^{\frac{n}{2}}(\frac{1}{2} - t)^{n/2}} \qquad \alpha = \frac{n}{2}, \beta = \frac{1}{2}$$

$$\Longrightarrow \boxed{f_Z(x) = \frac{(1/2)^{\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{1}{2}x}}{\Gamma(\frac{n}{2})}} \qquad \left(\frac{1}{2}\right)^{\frac{n}{2}} = \frac{1}{2^{\frac{n}{2}}}$$

Q1 (d) from 2022 exam.

$f_{X,Y}(x,y) = c\, e^{-3x} \times e^{-5y}$ if $x, y \geq 0$.



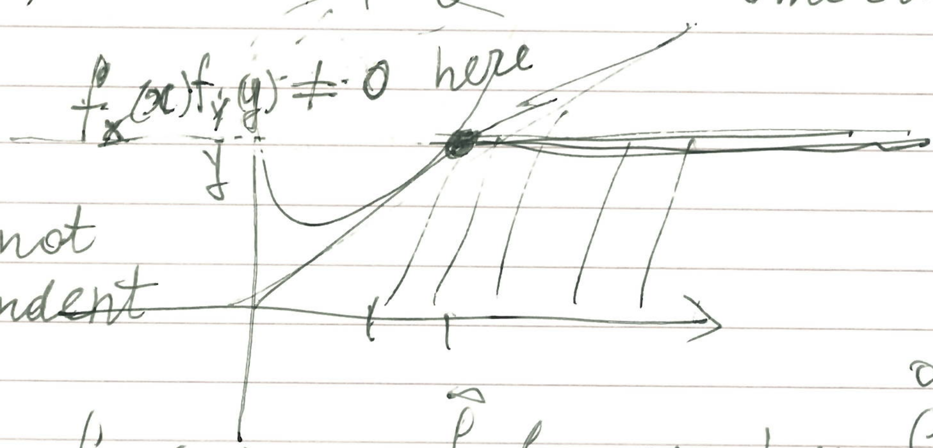
Since $f_{X,Y}(x,y) = h(x)\, g(y)$,

$$h(x) = \begin{cases} c\, e^{-3x}, & x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$g(y) = \begin{cases} e^{-5y}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0. \end{cases}$$

So, X, Y are independent

$$f_{X,Y}(x,y) = \begin{cases} d\, e^{-3x-5y}, & x \geq y \geq 0. \\ 0 & \text{otherwise.} \end{cases}$$



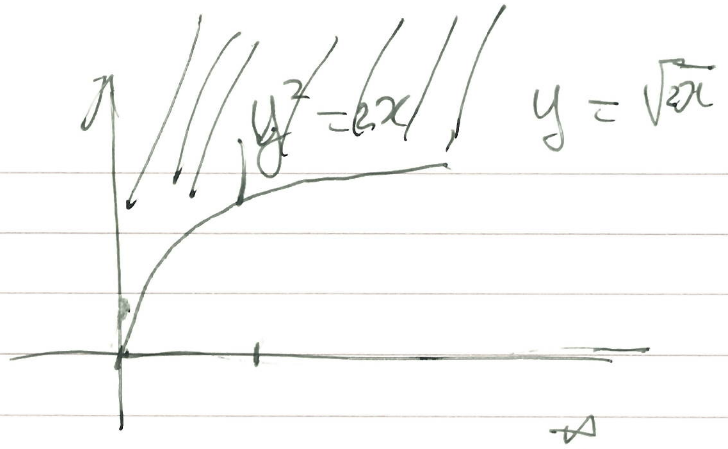
X, Y are not independent

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^{\infty} f_{X,Y}(x,y)\, dx$$

$f_Y(y) > 0$, $y > 0$.

$f_X(x) > 0$, if $x > 0$

$f_Y(y) f_X(x) > 0$.



$$\mathbb{P}(Y^2 \leq 2X) = \int_0^\infty dx \int_{\sqrt{2x}}^\infty f_{X,Y}(x,y)\,dy$$