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Problem Set 10 – Solutions.

1) a) i) $\int_{|z|=1} \frac{e^z \cos \pi z}{z^2+2z} dz$: We rewrite the integral as follows:

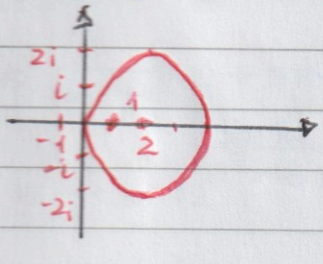
$$\int_{|z|=1} \frac{e^z \cos\pi z}{z^2+2z} dz = \int_{|z|=1} \frac{e^z \cos\pi z}{z(z+2)} dz = \int_{|z|=1} \frac{1}{z} \cdot \frac{e^z \cos\pi z}{z+2} dz$$

The function $\frac{e^z \cos\pi z}{z+2}$ is analytic in $\{z : |z| \leq 1\}$, thus we can use the CIF and obtain:

$$\int_{|z|=1} \frac{1}{z} \cdot \frac{e^z \cos\pi z}{z+2} dz = 2\pi i \left[\frac{e^z \cos\pi z}{z+2}\right]\Bigg|_{\underbrace{z=0}} = 2\pi i \cdot \frac{1}{2} = \pi i$$

the only point in $|z| \leq 1$, where the denominator $= 0$

ii) $\int_{|z-2|=2} \frac{\cosh z}{z^4-1} dz$: $z^4 - 1 = (z^2-1)(z^2+1) = (z-1)(z+1)(z-i)(z+i)$



$\Rightarrow z^4 - 1 = 0 \iff z = 1$ or $z = -1$ or $z = i$ or $z = -i$

$|1-2| = 1 < 2 \Rightarrow 1 \in \{z \in \mathbb{C} : |z-2| \leq 2\}$

$|-1-2| = 3 > 2 \Rightarrow -1 \notin \{z \in \mathbb{C} : |z-2| \leq 2\}$

$|i-2| = \sqrt{5} > 2 \Rightarrow i \notin \{z \in \mathbb{C} : |z-2| \leq 2\}$

$|-i-2| = \sqrt{5} > 2 \Rightarrow -i \notin \{z \in \mathbb{C} : |z-2| \leq 2\}$

$\Rightarrow$ The only root of the denominator that is in $|z-2| < 2$ is $z = 1$.

We rewrite the integral:

$$\int_{|z-2|=2} \frac{\cosh z}{z^4-1} dz = \int_{|z-2|=2} \frac{1}{z-1} \cdot \frac{\cosh z}{(z+1)(z^2+1)} dz = \int_{|z-2|=2} \frac{f(z)}{z-1} dz,$$

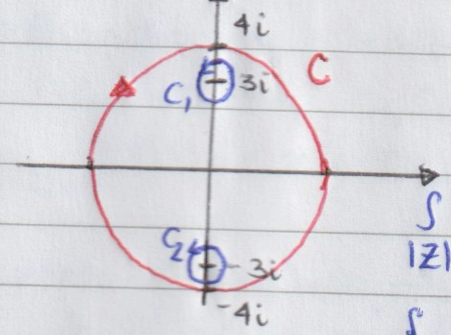
where $f(z) = \frac{\cosh z}{(z+1)(z^2+1)}$ is analytic in $|z-2| \leq 2$, thus by the CIF

$$\int_{|z-2|=2} \frac{f(z)}{z-1} dz = 2\pi i f(1) = 2\pi i \frac{\cosh 1}{2 \cdot 2} = \frac{\pi i}{2} \cosh 1$$

iii) $\int_{|z|=4} \frac{dz}{(z^2+9)(z+9)} = \int_{|z|=4} \frac{dz}{(z-3i)(z+3i)(z+9)}$

The denominator vanishes at $z = \pm 3i, -9$, namely only $z = \pm 3i$ are inside $|z| \leq 4$, thus we cannot apply CIF as is. The integral can be computed in several ways, for example: first, we apply the extended Deformation Principle (Cor 18.6) and then CIF as follows.

Encircle the points $z = \pm 3i$ by circular contours C_1 and C_2 so that they do not intersect and they are in the interior of the domain bounded by $|z| = 4$:



In the domain bounded by $|z|=4$, c_1, and c_2, the function $f(z)=\frac{1}{(z^2+9)(z+9)}$ is analytic, thus by Cor 18.6:

$$\int_{|z|=4} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

{ Choose for instance radius $\frac{1}{2}$ for C_1, C_2 }

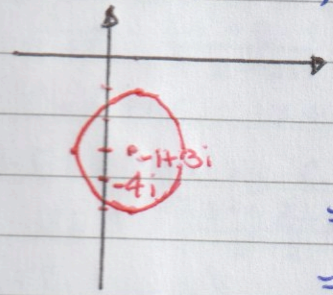
C_1, C_2 are positively oriented. For each of the integrals on the RHS we can apply CIF and get:

$$\int_{|z|=4} f(z)dz = 2\pi i\left[\frac{1}{(z+3i)(z+9)}\right]\Big|_{z=3i} + 2\pi i\left[\frac{1}{(z-3i)(z+9)}\right]\Big|_{z=-3i}$$

$$= 2\pi i\,\frac{1}{6i(9+3i)} + 2\pi i\,\frac{1}{-6i(9-3i)} = \frac{\pi}{3}\left[\frac{1}{9+3i}-\frac{1}{9-3i}\right] = -i\,\frac{\pi}{45}$$

iv) $$\int_{|z-1+3i|=2}\frac{\sin z}{z^3+16z}dz = \int_{|z-1+3i|=2}\frac{\sin z}{z(z+4i)(z-4i)}dz$$

There are 3 singularities: $z=0$ (removable - CHECK!) and $z=\pm 4i$ (simple poles - CHECK!) Since $|4i-1+3i|=\sqrt{50}>2$ and $|-4i-1+3i|=\sqrt{2}<2$ the only singularity inside $|z-1+3i|\le 2$ is $z=-4i$. Thus, by CIF we obtain



$$\int_{|z-1+3i|=2}\frac{1}{z+4i}\cdot\frac{\sin z}{z(z-4i)}dz \underset{\downarrow}{=} 2\pi i\left(\frac{\sin z}{z(z-4i)}\right)\Big|_{z=-4i}=$$

$\left\{\frac{\sin z}{z(z-4i)}\text{ is analytic in } |z-1+3i|\le 2\right\}$

$$= 2\pi i\,\frac{\sin(-4i)}{-4i(-8i)} = 2\pi i\,\frac{-\sin 4i}{-32} \underset{\downarrow}{=} \frac{\pi i}{16}\, i\sinh 4 =$$

$$= -\frac{\pi}{16}\sinh 4 \qquad \{\forall x\in\mathbb{R}:\ \sin(ix)=i\sinh x\}$$

v) $\int_{|z|=1}\frac{z^4+1}{z^2-2iz}dz = \int_{|z|=1}\frac{z^4+1}{z(z-2i)}dz$: the roots of the denominator

are $z=0, 2i$, only $z=0$ is inside $|z|\le 1$. The function $\frac{z^4+1}{z-2i}$ is analytic in $|z|\le 1$, thus by CIF we get

$$\int_{|z|=1}\frac{1}{z}\cdot\frac{z^4+1}{z-2i}dz = 2\pi i\left(\frac{z^4+1}{z-2i}\right)\Big|_{z=0} = 2\pi i\cdot\frac{1}{-2i} = -\pi$$

vi) $\int_{|z-2|=5}\frac{e^{z^2}}{z^2-6z}dz = \int_{|z-2|=5}\frac{e^{z^2}}{z(z-6)}dz$: the roots of the denominator are $z=0,6$ and since $|0-2|=2<5$, $|6-2|=4<5$ both roots are inside $|z-2|<5$. We continue in the same way as in iii): encircle $z=0$ and $z=6$ by circular contours C_1, C_2

(both positively oriented; of radius, for example $\frac{1}{2}$) such that $C_1 \cap C_2 = \emptyset$ and C_1, C_2 lie interior to $|z-2|=5$. By Cor 18.6

$$\int_{|z-2|=5} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

To each of the integrals on the RHS we apply CIF (justify!)

$$\int_{C_1} \frac{1}{z} \frac{e^{z^2}}{z-6}\,dz = 2\pi i \left(\frac{e^{z^2}}{z-6}\right)\Big|_{z=0} = 2\pi i \frac{1}{-6} = -\frac{\pi i}{3}$$

$$\int_{C_2} \frac{1}{z-6} \cdot \frac{e^{z^2}}{z}\,dz = 2\pi i \left(\frac{e^{z^2}}{z}\right)\Big|_{z=6} = 2\pi i \frac{e^{36}}{6} = \frac{e^{36}}{3}\pi i$$

$$\Rightarrow \int_{|z-2|=5} \frac{e^{z^2}}{z(z-6)}\,dz = \frac{\pi i}{3}(e^{36}-1)$$

b) $\int_C \frac{2z^2+z+1}{z^2+4}\,dz$ – the roots of the denominator $z = \pm 2i$; the numerator at $\pm 2i = -7 \pm 2i \neq 0$

i) $C = \{z \in \mathbb{C} : |z| = 1\}$: $\pm 2i$ are outside $C \Rightarrow f(z) = \frac{2z^2+z+1}{z^2+4}$ is analytic in $|z| \leq 1 \Rightarrow$ by Cauchy's Theorem (Thm 18.3) $\int_C f(z)dz = 0$

ii) $C = \{z \in \mathbb{C} : |z-2i| = 2\}$: Since $|2i-2i| = 0 < 2$ and $|-2i-2i| = 4 > 2$, $2i \in \{z \in \mathbb{C} : |z-2i| \leq 2\}$ and $-2i \notin \{z \in \mathbb{C} : |z-2i| \leq 2\}$, thus by CIF we obtain (since $\frac{2z^2+z+1}{z+2i}$ is analytic in $|z-2i| \leq 2$; $z = \pm 2i$ are both simple poles – WHY?)

$$\int_C \frac{1}{z-2i} \cdot \frac{2z^2+z+1}{z+2i}\,dz = 2\pi i \left(\frac{2z^2+z+1}{z+2i}\right)\Big|_{z=2i} = 2\pi i \frac{-7+2i}{4i} = \frac{\pi}{2}(-7+2i)$$

iii) $C = \{z \in \mathbb{C} : |z+i| = 2\}$: as in ii) only $-2i$ is inside $|z+i| \leq 2$ (CHECK) $\Rightarrow$ by CIF we obtain

$$\int_C \frac{1}{z+2i} \cdot \frac{2z^2+z+1}{z-2i}\,dz = 2\pi i \left(\frac{2z^2+z+1}{z-2i}\right)\Big|_{z=-2i} = 2\pi i \frac{-7-2i}{-4i} = +\frac{\pi}{2}(7+2i)$$

iv) $C = \{z : |z| = 4\}$: $z = \pm 2i$ both inside C (CHECK!). Continue as in 1 a) iii) – encircle $\pm 2i$ by C_1, C_2 (both positively oriented – what radius?), by Cor 18.6 combined with CIF:

$$\int_{|z|=4} \frac{2z^2+z+1}{z^2+4}\,dz \underset{\text{Cor 18.6}}{=} \int_{C_1 \text{ (around } 2i)} \frac{1}{z-2i} \frac{2z^2+z+1}{z+2i}\,dz + \int_{C_2 \text{ (around } -2i)} \frac{1}{z+2i} \cdot \frac{2z^2+z+1}{z-2i}\,dz$$

$$\underset{\text{CIF (Justify!)}}{=} 2\pi i \left(\frac{2z^2+z+1}{z+2i}\right)\Big|_{z=2i} + 2\pi i \left(\frac{2z^2+z+1}{z-2i}\right)\Big|_{z=-2i} = \frac{\pi}{2}(-7+2i) +$$

$$+ \frac{\pi}{2}(+7+2i) = \frac{\pi}{2}(-7+2i+7+2i) = 2\pi i$$

2) a) Extended CIF: Cor 18.9

i) $\int_{|z-1|=1} \left(\frac{z^2+1}{z^2-1}\right)^3 dz$

The roots of the denominator are $z = \pm 1$ and only $z = 1$ is inside $|z-1| \leq 1$ (CHECK!)
$z = \pm 1$ are simple poles, since $f(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z-1)(z+1)}$ => for $z=1$ $f(z) = \frac{z^2+1}{z+1} \cdot \frac{1}{z-1} = \varphi(z) \frac{1}{z-1}$, where $\varphi(z)$ is analytic at a neigborhood of $z=1$ (and at $z=1$) and $\varphi(1) = 1 \neq 0$ (in the same way $z=-1$ is a simple pole). Thus, we can apply Cor 18.9.

$$\int_{|z-1|=1} \left(\frac{z^2+1}{z^2-1}\right)^3 dz = \int_{|z-1|=1} \frac{1}{(z-1)^3} \left(\frac{z^2+1}{z+1}\right)^3 dz = \int_{|z-1|=1} \frac{1}{(z-1)^3} g(z)\,dz,$$

where $g(z) = \left(\frac{z^2+1}{z+1}\right)^3$ is analytic in $|z-1| \leq 1$ => by Cor 18.9 with $n=2$ we obtain

$$g^{(2)}(1) = \frac{2!}{2\pi i} \int_{|z-1|=1} \frac{1}{(z-1)^3} g(z)\,dz \Rightarrow \int_{|z-1|=1} \frac{g(z)}{(z-1)^3} dz = g''(1) \frac{2\pi i}{2!} = g''(1)\pi i$$

$$g'(z) = 3\left(\frac{z^2+1}{z+1}\right)^2 \frac{z^2+2z-1}{(z+1)^2} \qquad g''(z) = 6 \frac{z^2+1}{(z+1)^5}\left((z^2+2z-1)^2 + 2(z^2+1)\right)$$

=> $g''(1) = 3$, and

$$\int_{|z-1|=1} \frac{g(z)}{(z-1)^3} dz = 3\pi i$$

ii) $\int_{|z|=2} \frac{e^{iz}}{(z+i)^5}$: the only root of the denominator is $z=-i$ - it is a pole of order 5 (CHECK!) and it is inside $|z| \leq 2$.
Set $g(z) = e^{iz}$ and apply Cor 18.9 with $n=4$:

$$g^{(4)}(-i) = \frac{4!}{2\pi i} \int_{|z|=2} \frac{g(z)}{(z+i)^5} dz \Rightarrow \int_{|z|=2} \frac{g(z)}{(z+i)^5} dz = g^{(4)}(-i) \frac{2\pi i}{4!}$$

$g'(z) = ie^{iz}$ $\quad g''(z) = -e^{iz}$ $\quad g'''(z) = -ie^{iz}$ $\quad g^{(4)}(z) = e^{iz}$ => $g^{(4)}(-i) = e$

$$\Rightarrow \int_{|z|=2} \frac{g(z)}{(z+i)^5} dz = e \cdot \frac{\pi i}{12}$$

iii) $\int_{|z|=3} \frac{z^4}{(z+1)^3} dz$: $f(z) = \frac{z^4}{(z+1)^3}$ has a pole of order 3 at $z=-1$ (CHECK!) and $-1 \in |z| \leq 3$. Set $g(z) = z^4$ and apply Cor 18.9 with $n=2$ ($g(z)$ is analytic in $|z| \leq 3$):

$$g''(-1) = \frac{2!}{2\pi i} \int_{|z|=3} \frac{g(z)}{(z+1)^3} dz = \frac{1}{\pi i} \int_{|z|=3} \frac{g(z)}{(z+1)^3} dz \Rightarrow \int_{|z|=3} \frac{g(z)}{(z+1)^3} dz = \pi i\, g''(-1)$$

$g'(z) = 4z^3$, $g''(z) = 12z^2$ => $g''(1) = 12$ => $\int_{|z|=3} \frac{g(z)}{(z+1)^3} dz = 12\pi i$

iv) $\int_C \frac{e^z}{z(1-z)^3} dz$: $f(z) = \frac{e^z}{z(1-z)^3}$ has a simple pole at $z=0$

and a pole of order 3 at $z=1$ (CHECK!) There are 4 cases:

1) $z=0$ and $z=1$ are both outside C. Then, by Cauchy's Theorem (Thm 18.3) the integral equals to 0.

2) $z=0$ is inside C and $z=1$ is outside. Namely, $\frac{e^z}{(1-z)^3}$ is analytic inside C and since $z=0$ is a simple pole, by CIF:

$$\int_C \frac{1}{z}\frac{e^z}{(1-z)^3}\,dz = 2\pi i\left(\frac{e^z}{(1-z)^3}\right)\Big|_{z=0} = 2\pi i$$

3) $z=1$ is inside C and $z=0$ is outside $\Rightarrow \frac{e^z}{z}$ is analytic inside (and on) C $\Rightarrow$ by Cor 18.9 with $n=2$ (since $z=1$ is a pole of order 3) for $f(z)=\frac{e^z}{z}$

$$f''(1) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(1-z)^3}\,dz = \frac{1}{\pi i}\int_C \frac{f(z)}{(1-z)^3}\,dz \Rightarrow \int_C \frac{f(z)}{(1-z)^3}\,dz = \pi i f''(1)$$

$f'(z) = \frac{e^z(z-1)}{z^2}$ $\quad f''(z) = \frac{e^z}{z^3}(z^2-2z+2) \Rightarrow f''(1) = e$, and

$$\int_C \frac{f(z)}{(1-z)^3}\,dz = e\pi i$$

4) Both $z=0$ and $z=1$ are inside C. Then, as in 1 a) iii):

$$\int_C \frac{e^z}{z(1-z)^3}\,dz \underset{\text{Cor 18.6}}{=} \int_{C_1 \text{(encircles } z=0)} \frac{e^z}{z(1-z)^3}\,dz + \int_{C_2 \text{(encircles } z=1)} \frac{e^z}{z(1-z)^3}\,dz = \pi i(2+e)$$

b) We have $\int_C \frac{z^3-2z}{(z-z_0)^3}\,dz$. Since $z_0 \neq 0, \pm\sqrt{2}$, z_0 is a pole of order 3 of $\frac{z^3-2z}{(z-z_0)^3}$ (justify!), therefore, since z_0 is the only singularity of $f(z)$, if it is outside of D, then $f(z)=\frac{z^3-2z}{(z-z_0)^3}$ is analytic in D and by Cauchy's Theorem the integral is 0.
If $z_0 \in D$ we apply Cor 18.9 with $n=2$ and get:
Set $g(z) = z^3-2z$. Then

$$g''(z_0) = \frac{2!}{2\pi i}\int_C \frac{g(z)}{(z-z_0)^3}\,dz = \frac{1}{\pi i}\int_C \frac{g(z)}{(z-z_0)^3}\,dz \Rightarrow \int_C \frac{g(z)}{(z-z_0)^3}\,dz = \pi i g''(z_0)$$

$g'(z) = 3z^2-2$, $\quad g''(z) = 6z \Rightarrow g''(z_0) = 6z_0 \Rightarrow \int_C \frac{g(z)}{(z-z_0)^3}\,dz = 6\pi i z_0.$

What if $z_0=0$ or $z_0=\sqrt{2}$ or $z_0=-\sqrt{2}$?

3) a) i) $\frac{\sin z}{z^2}$: the singularity is at $z=0$. We look at the Laurent series expansion (about $z=0$):

$$\frac{1}{z^2}\sin z = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \dots$$

$\Rightarrow$ the coefficient of $\frac{1}{z-0} = \frac{1}{z}$ is $1 \Rightarrow \operatorname{Res}\left(\frac{\sin z}{z^2}, 0\right) = 1$

ii) $\frac{e^{z^2}}{z^{2n+1}}$: the only singularity is at $z=0$. We look at the Laurent series expansion of e^{z^2} (about $z=0$):

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}, \text{ therefore}$$

$$\frac{1}{z^{2n+1}}e^{z^2} = \frac{1}{z^{2n+1}}\left(1+z^2+\frac{z^4}{2!}+\dots+\frac{z^{2n}}{n!}+\frac{z^{2n+2}}{(n+1)!}+\dots\right) =$$

$= \frac{1}{z^{2n+1}} + \frac{1}{z^{2n-1}} + \frac{1}{2!\,z^{2n-3}} + \dots + \frac{1}{n!\,z} + \frac{z}{(n+1)!} + \dots$

Thus, the coefficient of $\frac{1}{z-0} = \frac{1}{z}$ is $\frac{1}{n!} \Rightarrow Res\left(\frac{e^{z^2}}{z^{2n+1}}, 0\right) = \frac{1}{n!}$

iii) $\frac{4z-1}{z^2+3z+2} = \frac{4z-1}{(z+2)(z+1)} \Rightarrow$ the singularities are $z=-2,-1$.

Both singularities are simple poles, since:

$z=-2$ $\quad \frac{4z-1}{z^2+3z+2} = \frac{1}{z+2} \cdot \frac{4z-1}{z+1} = \frac{1}{z+2}\varphi(z)$ and $\varphi(z)$ is analytic at a neighborhood of $z=-2$ (and at $z=-2$), and $\varphi(-2) = 9 \neq 0$

$z=-1$: $\quad \frac{4z-1}{z^2+3z+2} = \frac{1}{z+1} \cdot \frac{4z-1}{z+2} = \frac{1}{z+1} g(z)$ and $g(z)$ is analytic

at a neighborhood of $z=-1$ (and at $z=-1$), and $g(-1) = -5 \neq 0$

$\Rightarrow$ by Prop 16.2 $z=-2$ and $z=-1$ are both simple poles.

$\Rightarrow$ by Prop 16.2: $Res\left(\frac{4z-1}{z^2+3z+2}, -2\right) = 9$, $Res\left(\frac{4z-1}{z^2+3z+2}, -1\right) = -5$

iv) $\frac{1}{z+z^3} = \frac{1}{z(1+z^2)} = \frac{1}{z(z+i)(z-i)} \Rightarrow$ the singularities are at $z = 0, \pm i$, all simple poles (CHECK!). By Prop 16.2:

$z=0$: $\frac{1}{z+z^3} = \frac{1}{z} g(z)$, $g(z) = \frac{1}{z^2+1}$, $g(0) = 1 \neq 0 \Rightarrow Res\left(\frac{1}{z+z^3}, 0\right) = 1$

$z=i$: $\frac{1}{z+z^3} = \frac{1}{z-i}\tilde{g}(z)$, $\tilde{g}(z) = \frac{1}{z(z+i)}$, $\tilde{g}(i) = -\frac{1}{2} \Rightarrow Res\left(\frac{1}{z+z^3}, i\right) = -\frac{1}{2}$

$z=-i$: $\frac{1}{z+z^3} = \frac{1}{z+i}\varphi(z)$, $\varphi(z) = \frac{1}{z(z-i)}$, $\varphi(-i) = -\frac{1}{2} \Rightarrow Res\left(\frac{1}{z+z^3}, -i\right) = -\frac{1}{2}$

vi) $\frac{\sin \pi z}{(z-1)^3}$: The singularity is at $z=1$, but note: $\sin \pi \cdot 1 = 0$!
To determine the nature of the singularity we look at the Taylor series expansion of $\sin \pi z$ around $z=1$! First, we look for the coefficients:

$f(z) = \sin \pi z \qquad f'(z) = \pi \cos \pi z \qquad f''(z) = -\pi^2 \sin \pi z \qquad f^{(3)}(z) = -\pi^3 \cos \pi z$

$f(1) = 0 \qquad f'(1) = -\pi \qquad f''(1) = 0 \qquad f^{(3)}(1) = \pi^3$

and so on. Therefore,

$\sin \pi z = 0 + (-\pi)(z-1) + \frac{\pi^3}{3!}(z-1)^3 - \frac{\pi^5}{5!}(z-1)^5 + \dots$

$\Rightarrow \frac{\sin \pi z}{(z-1)^3} = \frac{1}{(z-1)^3}\left(-\pi(z-1) + \frac{\pi^3}{3!}(z-1)^3 - \frac{\pi^5}{5!}(z-1)^5 + \dots\right)$

$= -\frac{\pi}{(z-1)^2} + \frac{\pi^3}{3!} - \frac{\pi^5}{5!}(z-1)^2 + \dots$

$\Rightarrow$ the coefficient of the term $\frac{1}{z-1}$ is $0 \Rightarrow Res\left(\frac{\sin \pi z}{(z-1)^3}, 1\right) = 0$

v) $\frac{\cos z}{(z-1)^2}$: the singularity is at $z=1$ and it is a pole of order 2 since $\frac{\cos z}{(z-1)^2} = \frac{1}{(z-1)^2}\varphi(z)$, $\varphi(z) = \cos z$ is analytic at a neighborhood of $z=1$ (and at $z=1$), $\varphi(1)=\cos 1 \neq 0 \Rightarrow$ by Prop. 16.2 $z=1$ is a pole of order 2. By Prop 16.2:

$$\operatorname{Res}\left(\frac{\cos z}{(z-1)^2}, 1\right) = \frac{\varphi'(1)}{1!} = -\sin 1.$$

b) i) [State the Residue Theorem if you use it!]

$$\int_{|z|=1} \frac{z^2+1}{z^2-2z}\,dz = \int_{|z|=1} \frac{z^2+1}{z(z-2)}\,dz = \int_{|z|=1} \frac{1}{z}\cdot\frac{z^2+1}{z-2}\,dz$$

There are 2 singularities: $z=0$, $z=2$, but $z=2$ is outside $|z|\leq 1$, thus does not contribute to the value of the integral. $z=0$ is a simple pole (Justify!) $\Rightarrow$ by Prop 16.2 (formulate!) $\operatorname{Res}\left(\frac{z^2+1}{z^2-2z}, 0\right) = -\frac{1}{2}$. Thus, by the Residue Theorem (first, explain why the conditions of the Residue Theorem are satisfied!)

$$\int_{|z|=1} \frac{z^2+1}{z^2-2z}\,dz = 2\pi i \operatorname{Res}\left(\frac{z^2+1}{z^2-2z}, 0\right) = 2\pi i\left(-\frac{1}{2}\right) = -\pi i.$$

ii) $\int_{|z|=1} \frac{1}{z} e^{\frac{1}{z}}\,dz$ - the only singularity is at $z=0$. To find the residue we write the Laurent series expansion for $e^{\frac{1}{z}}$ (about $z=0$):

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \Rightarrow \frac{1}{z}e^{\frac{1}{z}} = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\dots\right) = \frac{1}{z}+\frac{1}{z^2}+\dots$$

$\Rightarrow z=0$ is an essential singularity (Why?) and $\operatorname{Res}\left(\frac{e^{\frac{1}{z}}}{z}, 0\right) = 1$ $\Rightarrow$ by the Residue Theorem ($\frac{1}{z}e^{\frac{1}{z}}$ is analytic on and inside $|z|=1$, that is simple, closed, positively oriented contour, except at $z=0$, that is an isolated singularity - essential, which is inside the contour $|z|=1 \Rightarrow$ all the conditions of the Residue Theorem hold and we can apply it)

$$\int_{|z|=1} \frac{1}{z} e^{\frac{1}{z}}\,dz = 2\pi i \cdot 1 = 2\pi i$$

iii) $\int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z+1)^2}$: there are 2 singularities $z=\pm 1$, both are poles of order 2 (Justify!). Since $|-1-1-i| = \sqrt{5} > 2$ and $|1-1-i| = |-i| = 1 < 2$ only $z=1$ is inside $|z-1-i| \leq 2$.

By Prop 16.2: $\operatorname{Res}\left(\frac{1}{(z-1)^2(z+1)^2}, 1\right) = \frac{1}{1!}\left(\frac{1}{(z+1)^2}\right)'_{z=1} = -\frac{1}{4}$.

Thus, by the Residue Theorem (why applicable?)

$$\int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z+1)^2} = 2\pi i \operatorname{Res}\left(\frac{1}{(z-1)^2(z+1)^2}, 1\right) = -\frac{\pi i}{2}.$$

iv) $\int_{|z|=2} \frac{\sin z}{(z+1)^3} dz$: the only singularity is $z=-1$ and it is a pole of order 3 (Justify!). By Prop 16.2 for $\varphi(z) = \sin z$:

$$\text{Res}\left(\frac{\varphi(z)}{(z+1)^3}, -1\right) = \frac{\varphi''(-1)}{2!} = \frac{-\sin(-1)}{2!} = \frac{\sin 1}{2}$$

$\Rightarrow$ by Residue Theorem (Why applicable?), since $z=-1$ is inside $|z| \leq 2$

$$\int_{|z|=2} \frac{\sin z}{(z+1)^3} dz = 2\pi i \,\text{Res}\left(\frac{\sin z}{(z+1)^3}, -1\right) = 2\pi i \frac{\sin 1}{2} = \pi i \sin 1.$$

v) We compute this integral in 2 ways: by the application of the Residue Theorem and by using the extended Cauchy Integral Formula.

1) The singularities of $f(z) = \frac{1}{(z-5)(z+1)^4}$ are at $z=5$ and $z=-1$ since $|5-1| = 4 > \frac{5}{2}$ it is outside of $|z-1| \leq \frac{5}{2}$, thus does not contribute to the value of the integral. $|-1-1| = 2 < \frac{5}{2}$ is inside $|z-1| \leq \frac{5}{2}$ and $z=-1$ is a pole of order 4 (Justify!)
By Prop 16.2 with $\varphi(z) = \frac{1}{z-5}$:

$$\text{Res}(f; -1) = \frac{\varphi^{(3)}(-1)}{3!} = \frac{1}{6}\left(-\frac{1}{216}\right) = -\frac{1}{6^4},$$

since $\varphi'(z) = -\frac{1}{(z-5)^2}$, $\varphi''(z) = \frac{2}{(z-5)^3}$, $\varphi^{(3)}(z) = -\frac{6}{(z-5)^4}$

$\Rightarrow$ by the Residue Theorem (why it is applicable?)

$$\int_{|z-1|=\frac{5}{2}} f(z) = 2\pi i \,\text{Res}(f; -1) = -\frac{2\pi i}{6^4}$$

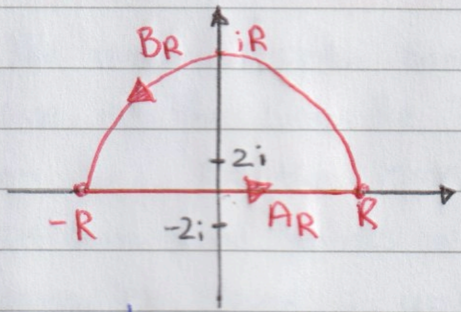
2) By extended Cauchy Formula (why it is applicable?) for $\varphi(z) = \frac{1}{z-5}$ and $n=3$:

$$\varphi^{(3)}(-1) = \frac{3!}{2\pi i} \int_{|z-1|=\frac{5}{2}} \frac{\varphi(z)}{(z+1)^4} dz \Rightarrow \int_{|z-1|=\frac{5}{2}} \frac{\varphi(z)}{(z+1)^4} dz = \frac{2\pi i}{3!} \varphi^{(3)}(-1)$$
$$= -\frac{2\pi i}{6^4}$$

4) We evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$ by calculating $\int_C \frac{dz}{(z^2+4)^2}$ on an appropriate contour C and taking an appropriate limit:
Consider the contour integral:

$$\int_{C_R} \frac{dz}{(z^2+4)^2} = \int_{A_R} \frac{dz}{(z^2+4)^2} + \int_{B_R} \frac{dz}{(z^2+4)^2},$$

where $C_R = A_R \cup B_R$ is the contour given by:



Since $z^2+4=(z-2i)(z+2i)$ the contour encloses the pole of order 2 (why?) at $z=2i$ for any $R>2$.
Thus, by the Residue Theorem (Why it is applicable?) for any $R>2$

$$\int_{C_R} \frac{dz}{(z^2+4)^2} = \int_{C_R} \frac{1}{(z+2i)^2} \cdot \frac{1}{(z-2i)^2} dz = \int_{C_R} \frac{\varphi(z)}{(z-2i)^2} dz =$$

$$= 2\pi i \operatorname{Res}\left(\frac{\varphi(z)}{(z-2i)^2}, 2i\right) \underset{\text{Prop 16.2}}{=} 2\pi i \frac{\varphi'(2i)}{1!} \underset{\downarrow}{=} 2\pi i \left(-\frac{2}{(4i)^3}\right) = \frac{\pi}{16}$$

$$\varphi(z) = \frac{1}{(z+2i)^2} \Rightarrow \varphi'(z) = -\frac{2}{(z+2i)^3}$$

Next, observe that on B_R (semicircular contour), $|z|=R$, so by ML inequality, we get:

$$\left|\int_{B_R} \frac{dz}{(z^2+4)^2}\right| \le \underbrace{\pi R}_{\text{Length}(B_R)} \max_{|z|=R} \left|\frac{1}{(z^2+4)^2}\right| \underset{(*)}{\le} \frac{\pi R}{(R^2-4)^2}$$

$(*)$ $|(z^2+4)^2| = |z^2+4|^2 \underset{\text{reverse Triangle inequality}}{\ge} ||z|^2-4|^2 = |R^2-4|^2 \underset{R>2}{=} (R^2-4)^2$

Thus: $0 \le \lim_{R\to\infty} \left|\int_{B_R} f(z)dz\right| \le \lim_{R\to\infty} \frac{\pi R}{(R^2-4)^2} = 0.$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \lim_{R\to\infty} \int_{C_R} f(z)dz = \frac{\pi}{16} + 0 = \frac{\pi}{16} \quad \blacksquare.$$