Problem Set 10.

1) a) Using the Cauchy Integral formula (or otherwise) compute the following integrals

i) $\int_{|z|=1} \frac{e^z \cos \pi z}{z^2+2z}\,dz$ ii) $\int_{|z-2|=2} \frac{\cosh z}{z^4-1}\,dz$ iii) $\int_{|z|=4} \frac{dz}{(z^2+9)(z+9)}$

iv) $\int_{|z-1+3i|=2} \frac{\sin z}{z^3+16z}\,dz$ v) $\int_{|z|=1} \frac{z^4+1}{z^2-2iz}\,dz$ vi) $\int_{|z-2|=5} \frac{e^{z^2}}{z^2-6z}\,dz$

b) Compute $\int_C \frac{2z^2+z+1}{z^2+4}\,dz$, where

i) $C=\{z: |z|=1\}$ ii) $C=\{z: |z-i|=2\}$ iii) $C=\{z: |z+i|=2\}$
iv) $C=\{z: |z|=4\}$ (Justify your steps!)

2) a) Using the extended CIF compute: (Justify your steps!)

i) $\int_{|z-1|=1} \left(\frac{z^2+1}{z^2-1}\right)^3 dz$ ii) $\int_{|z|=2} \frac{e^{iz}}{(z+i)^5}\,dz$ iii) $\int_{|z|=3} \frac{z^4}{(z+1)^3}\,dz$

iv) $\int_C \frac{e^z}{z(1-z)^3}\,dz$ – C is a simple, closed, positively oriented contour that does not pass through $z=0$, $z=1$. Study all cases!

b) Let D be a region bounded by a simple, closed, positively oriented curve C. Prove: $\int_C \frac{z^3-2z}{(z-z_0)^3}\,dz = \begin{cases} 6\pi i z_0 & z_0\in D \\ 0 & z_0 \notin D \end{cases}$
$(z_0 \neq 0, \pm\sqrt{2})$

3) a) Compute the residue of the following functions at <u>all</u> singular points:

i) $\frac{\sin z}{z^2}$ ii) $\frac{e^{z^2}}{z^{2n+1}}$ iii) $\frac{4z-1}{z^2+3z+2}$ iv) $\frac{1}{z+z^3}$ v) $\frac{\cos z}{(z-1)^2}$ vi) $\frac{\sin \pi z}{(z-1)^3}$

b) Using the Residue Theorem (or otherwise) compute the following integrals (all the curves are positively oriented; state all the theorems/propositions that are used in the solution)

i) $\int_{|z|=1} \frac{z^2+1}{z^2-2z}\,dz$ ii) $\int_{|z|=1} \frac{1}{z} e^{\frac{1}{z}}\,dz$ iii) $\int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z+1)^2}$

iv) $\int_{|z|=2} \frac{\sin z}{(z+1)^3}\,dz$ v) $\int_{|z-1|=\frac{5}{2}} \frac{dz}{(z-5)(z+1)^4}$

4) Using the Residue Theorem, show that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{16}$