Week 11. Lecture 28.

We have proved Cauchy's theorem and have seen an important corollary of it - the Deformation Principle. We are about to see more important consequences of Cauchy's Theorem.

Thm 18.7 (The Cauchy Integral Formula): If f is analytic on and everywhere inside a simple, closed, positively oriented contour C, and if z_0 is any point inside C, then

$$\text{CIF:} \quad f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

In other words: once we know the value of f at every point <u>on</u> C, we can compute its value at every point <u>inside</u> C.

<u>Pf</u>: Let $\varepsilon>0$. By Prop 6.5 f is continuous $\Rightarrow$ there exists $\delta>0$ such that if $|z-z_0|<\delta$, then $|f(z)-f(z_0)|<\varepsilon$. Choose $R>0$ such that $R<\delta$ and such that the disc of radius R centered at z_0 is inside C. Let C_0 be the positively oriented boundary of that disc, parametrized by $\gamma(t) = z_0 + Re^{it}$, $0\le t\le 2\pi$. We have:

$$\int_C \frac{f(z)}{z-z_0}\,dz \underset{\text{Deformation Principle}}{=} \int_{C_0}\frac{f(z)}{z-z_0}\,dz \underset{\text{add and subtract } f(z_0)}{=} \int_{C_0}\frac{f(z_0)}{z-z_0}\,dz +$$

$$+\int_{C_0}\frac{f(z)-f(z_0)}{z-z_0}\,dz = f(z_0)\int_0^{2\pi}\frac{Rie^{it}}{Re^{it}}\,dt + I$$

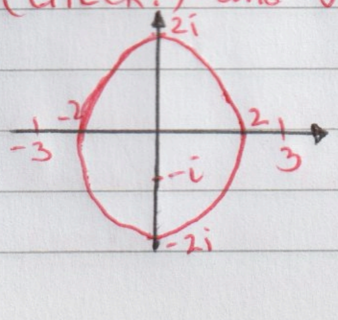
$$= f(z_0)\,2\pi i + I,$$

where: $|I| \underset{\text{Prop 17.5}}{\le} \max_{z\in C_0}\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \text{Length}(C_0) \underset{z\in C_0 \Rightarrow |z-z_0|=R}{<} \frac{\varepsilon}{R}\cdot 2\pi R = 2\pi\varepsilon$

But $\varepsilon>0$ is arbitrary, therefore we conclude that $I=0$ ∎

Ex 20.2: Compute $\int_C \frac{z}{(9-z^2)(z+i)}\,dz$, where C is the positively oriented circle centered at the origin of radius 2.

<u>Sol</u>: Note: $f(z) = \frac{z}{(9-z^2)(z+i)}$ has simple poles at $z=-i, \pm 3$ (CHECK!) and only $z=-i$ is enclosed by C. Therefore:



$$\int_C \frac{z}{(9-z^2)(z+i)}\,dz = \int_C \frac{f(z)}{z+i}\,dz,$$

where $f(z) = \frac{z}{9-z^2}$ is analytic on and everywhere inside C. By CIF (Thm 18.7):

$$\int_C \frac{f(z)}{z+i}\,dz = 2\pi i\, f(-i) = 2\pi i \frac{-i}{9-(-i)^2} = \frac{\pi}{5}.$$

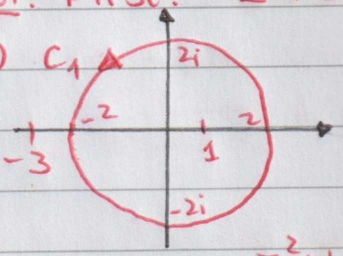
As this example suggests, there is a connection between the singularities of a given complex function $f(z)$ and the value of the contour integral for a given contour. In fact, there is an even stronger statement that one can make, relating the integral to the Laurent series expansion of f. We will not see it here.

Ex 20.3: Compute $\int_C \frac{z^2+1}{z^2+2z-3}\,dz$, where

a) $C_1 = \{z \in \mathbb{C} : |z| = 2\}$ c) $C_3 = \{z \in \mathbb{C} : |z+1| = 1\}$

b) $C_2 = \{z \in \mathbb{C} : |z+2+3i| = 4\}$ d) $C_4 = \{z \in \mathbb{C} : |z+i| = 5\}$

Sol: First: $z^2+2z-3=0 \iff z=1, z=-3$.

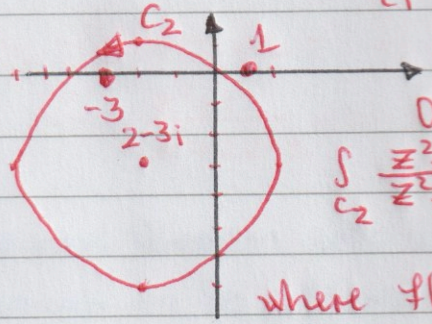
a) 

Inside C_1 the denominator is equal to 0 only at $z=1$. Rewrite:

$$\int_{C_1} \frac{z^2+1}{z^2+2z-3}\,dz = \int_{C_1} \frac{z^2+1}{z+3}\cdot\frac{1}{z-1}\,dz = \int_{C_1} \frac{f(z)}{z-1}\,dz,$$

where $f(z) = \frac{z^2+1}{z+3}$ is analytic on and everywhere inside C_1. Hence, by CIF:

$$\int_{C_1} \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i \tfrac{1}{2} = \pi i$$

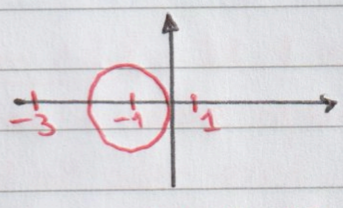
b) 

Inside C_2 the denominator is equal to 0 only at $z=-3$. Rewrite:

$$\int_{C_2} \frac{z^2+1}{z^2+2z-3}\,dz = \int_{C_2} \frac{z^2+1}{z-1}\cdot\frac{1}{z+3}\,dz = \int_{C_2} \frac{f(z)}{z+3}\,dz,$$

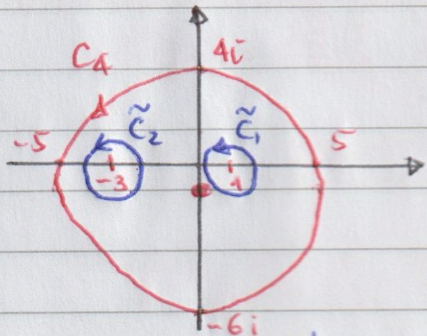
where $f(z) = \frac{z^2+1}{z-1}$ is analytic on and everywhere inside C_2. Hence, by CIF, we get:

$$\int_{C_2} \frac{f(z)}{z+3}\,dz = 2\pi i\, f(-3) = 2\pi i \cdot \frac{10}{-4} = -5\pi i$$

c) 

The function $f(z) = \frac{z^2+1}{z^2+2z-3}$ is analytic on and everywhere inside C_3, therefore, by Cauchy's Theorem $\int_{C_3} f(z)\,dz = 0$

d) Since $|1+i| = \sqrt{2} < 5$, $|-3+i| = \sqrt{10} < 5$ both roots of the denominator lie in a domain bounded by C_4, therefore we cannot use CIF!



One can compute this integral by several different ways. We will use the generalized Deformation Principle and CIF as follows: encircle the points $z=1, z=-3$ by 2 circular contours $\tilde{C}_1, \tilde{C}_2$ so that they are disjoint and they are entirely lie interior to C_4: for example:

$$\tilde{C}_1 = \{z \in \mathbb{C} : |z-1| = 1\}, \quad \tilde{C}_2 = \{z \in \mathbb{C} : |z+3| = \tfrac{1}{2}\}$$

Then $\tilde{C}_1 \cap \tilde{C}_2 = \emptyset$ and $\tilde{C}_1, \tilde{C}_2$ lie interior to C_4. Then $f(z)$ is analytic in a domain that contains $C_4, \tilde{C}_1, \tilde{C}_2$, and the region between them. Therefore, by the generalized Deformation Principle (Cor 18.6):

$$\int_{C_4} \frac{z^2+1}{z^2+2z-3}\,dz = \int_{\tilde{C}_1} \frac{z^2+1}{z^2+2z-3}\,dz + \int_{\tilde{C}_2} \frac{z^2+1}{z^2+2z-3}\,dz,$$

$\tilde{C}_1$ and $\tilde{C}_2$ are positively oriented (anticlockwise)

For each of the integrals on the RHS we can use CIF:

$$\int_{\tilde{C}_1} \frac{z^2+1}{z^2+2z-3}\,dz = \int_{\tilde{C}_1} \frac{z^2+1}{z+3}\cdot\frac{1}{z-1}\,dz = \int_{\tilde{C}_1} \frac{f(z)}{z-1}\,dz = 2\pi i\, f(1) = \pi i$$

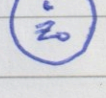
$$\int_{\tilde{C}_2} \frac{z^2+1}{z^2+2z-3}\,dz = \int_{\tilde{C}_2} \frac{z^2+1}{z-1}\cdot\frac{1}{z+3}\,dz = \int_{\tilde{C}_2} \frac{f(z)}{z+3}\,dz = 2\pi i\, f(-3) = -5\pi i$$

$$\Rightarrow \int_{C_4} \frac{z^2+1}{z^2+2z-3}\,dz = \pi i - 5\pi i = -4\pi i.$$

The next theorem tells us that for the complex function it is enough to be differentiable once to be differentiable infinitely many times - very different from the real functions! Once differentiable real functions need not be even differentiable twice, let alone infinitely many times!

Thm 18.8: If f is analytic at z_0 (f is differentiable at every point in some open disc centered at z_0), then the derivatives of f of all order exist and are analytic at z_0.

Pf (Sketch!) Choose a small, positively oriented circle around z_0 such that f is analytic on and everywhere inside it.



By CIF: $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$

Differentiating under the integral sign with respect to z_0, gives

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz$$

$$f^{(3)}(z_0) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^4}\,dz$$

$$\vdots$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Note: one cannot always differentiate under an integral in this manner. We should, in fact, prove a technical lemma, justifying why we are allowed to perform such an operation. For example, the first derivative - we need to interchange $\lim_{\Delta z \to 0}$ and the integral and this requires justification! ☒

Lectures 29+30

We have seen that once differentiable complex function is differentiable infinitely many times. A useful corollary is the following extended version of the Cauchy Integral Formula.

Cor 18.9 (Extended version of the CIF): Let f be analytic on and inside a simple, closed, positively oriented contour C and let z_0 be inside C. Then, for all $n \geq 0$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Ex 20.4: Compute $\int_C \frac{z^3}{(z-1)^4} dz$, where C is any simple, closed, positively oriented contour around $z=1$

Sol: Set $f(z) = z^3$, apply Cor 18.9 with $n=3$:

$$f^{(3)}(1) = \frac{3!}{2\pi i} \int_C \frac{f(z)}{(z-1)^4} dz \Rightarrow \int_C \frac{f(z)}{(z-1)^4} dz = f^{(3)}(1) \frac{2\pi i}{3!}$$

$f(z) = z^3 \Rightarrow f'(z) = 3z^2,\ f''(z) = 6z,\ f^{(3)}(z) = 6 \Rightarrow \int_C \frac{f(z)}{(z-1)^4} dz = 6 \cdot \frac{2\pi i}{6}$

Ex 20.5: Compute the following integrals

a) $\int_{|z|=1} \frac{\cos z}{z^3} dz$ b) $\int_{|z-3|=6} \frac{z}{(z-2)^2(z+5)} dz$

Sol: a) Set $f(z) = \cos z$ - it is analytic in $\{z \in \mathbb{C} : |z| \leq 1\}$, thus by Cor 18.9 with $n=2$, since $f'(z) = -\sin z$, $f''(z) = -\cos z$, $f''(0) = -1$

$$f^{(2)}(0) = \frac{2!}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^3} dz \Rightarrow \int_{|z|=1} \frac{f(z)}{z^3} dz = \pi i f''(0) = -\pi i$$

b) The function $f(z) = \frac{z}{z+5}$ is analytic in the disc $\{z \in \mathbb{C} : |z-3| \leq 6$

$\Rightarrow$ by Cor 18.9 with $n=1$, since $f'(z) = \frac{5}{(z+5)^2} \Rightarrow f'(2) = \frac{5}{49}$

$$f'(2) = \frac{1}{2\pi i} \int_{|z-3|=6} \frac{f(z)}{(z-2)^2} dz \Rightarrow \int_{|z-3|=6} \frac{f(z)}{(z-2)^2} dz = 2\pi i f'(2) = \frac{10\pi i}{49}$$

Now we are going to formulate and sketch a proof of another consequence of Cauchy's Theorem, that is a very powerful tool in computations of integrals.

Recall (Def 16.2): The coefficient b_1 of the term $\frac{1}{z-z_0}$ is called the residue of the singularity at $z = z_0$, denoted by $\mathrm{Res}(f, z_0)$.

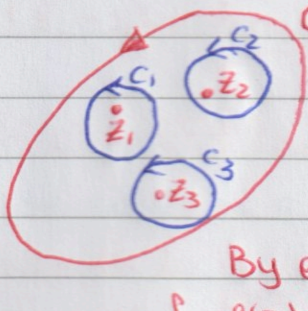
We have seen the definition and examples on week 8!

Now we are going to see how to compute integrals using only the values of the function at singularities - the residues.

Thm 19.1 (Residue Theorem) Let f be analytic on and inside a simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, .., z_n$ all inside C. Then,

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

Pf: We shall sketch the proof of the case where $z_1, .., z_n$ are simple poles or removable singularities.



Draw small positively oriented circles C_k around the singularities z_k such that they all inside C and they are disjoint - choose radius small enough, so it is possible.

By extended Deformation Principle (Cor 18.6):

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$$

If z_k is a removable singularity: By definition, f can be extended to an analytic function everywhere inside C_k, thus by Cauchy's Theorem $\int_{C_k} f(z) = 0$. But in this case $\text{Res}(f, z_k) = 0$ as well.

If z_k is a simple pole, then by Prop 16.2 $(m=1)$ $f(z) = \frac{\varphi(z)}{z - z_k}$ where $\varphi(z)$ is analytic at z_k, therefore by CIF (Thm 18.7)

$$\int_{C_k} f(z)dz = \int_{C_k} \frac{\varphi(z)}{z - z_k} dz = 2\pi i \varphi(z_k).$$

But by Prop 16.2 (for $m=1$) $\text{Res}(f, z_k) = \frac{\varphi(z_k)}{0!} = \varphi(z_k)$. ∎

Now we can work out examples applying the Residue Theorem.

Ex 21.1: Compute $\int_{|z|=1} \frac{\cos z}{z^5} dz$

Sol: Consider the Laurent series:

$$\frac{\cos z}{z^5} = \frac{1}{z^5}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) = \frac{1}{z^5} - \frac{1}{2! z^3} + \frac{1}{4! z} - \frac{z}{6!} + \dots$$

$$\Rightarrow b_1 = \frac{1}{4!} = \frac{1}{24} \quad \Rightarrow \int_{|z|=1} \frac{\cos z}{z^5} dz = 2\pi i \frac{1}{4!} = \frac{\pi i}{12}$$

Note: the singularity is $z=0$ that is a pole of order 4, and it is inside the disc $\{z \in \mathbb{C} : |z| \leq 1\}$. thus, the Residue Theorem is applicable.

Recall how we compute the residue of the poles. Start with the case of a simple pole

$z = z_0$ is a simple pole $\Rightarrow f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \dots$

(the Laurent series expansion)

Multiply both sides by $(z - z_0)$ and take the limit when z tends to z_0

$$\lim_{z \to z_0} (z - z_0) f(z) = b_1 = \text{Res}(f, z_0) \quad (*)$$

Ex 21.2: Compute $\mathrm{Res}(f, z_0)$ for $f(z) = \frac{3+4z}{(z+2)(z-i)}$ at $i, -2$.

Sol: $z_0 = i$ is a simple pole since $\left.\frac{3+4z}{z+2}\right|_{z=i} = \frac{3+4i}{i+2} \neq 0$ (and it is analytic at a neighborhood of i and at i) In the same way (CHECK) $z_0 = -2$ is a simple pole. Thus, from (*)

$$\mathrm{Res}(f,-2) = \lim_{z \to -2} (z+2)\frac{3+4z}{(z+2)(z-i)} = \left.\frac{3+4z}{z-i}\right|_{z=-2} = \frac{5}{2+i} = 2-i$$

$$\mathrm{Res}(f,i) = \left.\frac{3+4z}{z+2}\right|_{z=i} = \frac{3+4i}{i+2} = \frac{3+4i}{i+2}\cdot\frac{-i+2}{-i+2} = \frac{10+5i}{5} = 2+i$$

Now the case of a pole of order m.

By Prop 16.2 if z_0 is a pole of order m, then $\mathrm{Res}(f,z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$ where $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$, φ is analytic at a neighborhood of z_0 and at z_0. Thus, it is equivalent to:

(CHECK!) $\quad \mathrm{Res}(f,z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left[(z-z_0)^m f(z)\right]^{(m-1)}$ (**)

Ex 21.3: Compute $\mathrm{Res}(f;-2)$ for $f(z) = \frac{z+1}{(z-1)(z+2)^2}$

Sol: $\mathrm{Res}(f;-2) = \lim_{z\to-2}\left((z+2)^2 f(z)\right)' = \lim_{z\to -2}\left(\frac{z+1}{z-1}\right)' = \lim_{z\to-2}\frac{-2}{(z-1)^2} = -\frac{2}{9}$

(**) with $m=2$

Ex 21.4 Compute $\mathrm{Res}(f,0)$ for $f(z) = \frac{1}{z(e^z-1)}$

Sol: (CHECK!) $z=0$ is a pole of order 2.

$$z(e^z-1) = z\left(1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\dots-1\right) = z\left(z+\frac{z^2}{2}+\frac{z^3}{3!}+\dots\right)$$

$$= z^2\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)$$

$$\Rightarrow \mathrm{Res}(f;0) = \lim_{z\to 0}\left[z^2\cdot\frac{1}{z^2\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)}\right]' =$$

$$= \lim_{z\to 0}\frac{-\frac{1}{2}-\frac{2z}{3!}-\dots}{\left(1+\frac{z}{2}+\frac{z^2}{3!}+\dots\right)^2} = -\frac{1}{2}$$

Ex 21.5: Compute $\int_C \frac{z}{(z-4)^2}dz$, where C is the contour defined by traversing once the square with vertices $\pm 3i$, $6 \pm 3i$ anti-clockwise (See Ex 20.1 for another solution)

Sol: $z=4$ is a pole of order 2 of $\frac{z}{(z-4)^2}$ (Justify!) Other than at $z=4$ the function $\frac{z}{(z-4)^2}$ is analytic. Thus by Prop 16.2 for $\varphi(z) = z$: $\mathrm{Res}\left(\frac{z}{(z-4)^2}, 4\right) = \frac{\varphi'(4)}{1!} = 1$ $(\varphi'(z) = 1\ \forall z)$

$\Rightarrow$ By the Residue Theorem:

$$\int_C \frac{z}{(z-4)^2}dz = 2\pi i\,\mathrm{Res}\left(\frac{z}{(z-4)^2},4\right) = 2\pi i\,\frac{\varphi'(4)}{1!} = 2\pi i$$

Ex 21.6 Compute $\int_C \frac{dz}{(z-1)(z-2)}$ where C is some closed, simple, positively oriented contour.

Sol: $f(z) = \frac{1}{(z-1)(z-2)}$ has 2 simple poles $z=1, z=2$ (CHECK!)
$\operatorname{Res}(f,1) = -1$, $\operatorname{Res}(f,2) = 1$. Thus:

$$\int_C \frac{dz}{(z-1)(z-2)} = \begin{cases} -2\pi i & \text{if } C \text{ contains } z=1, \text{ but not } z=2 \\ 2\pi i & \text{if } C \text{ contains } z=2, \text{ but not } z=1 \\ 0 & \text{otherwise} \end{cases}$$

Ex 21.7: Compute $\int_{|z|=3} \frac{z+1}{(z-1)(z+2)^2}\,dz$

Sol: In the disc $\{z \in \mathbb{C} : |z| \le 3\}$ the function $f(z) = \frac{z+1}{(z-1)(z+2)^2}$ has 2 isolated singularities - poles at $z=1$ and at $z=-2$ (CHECK!) Therefore, by the Residue Theorem

$$\int_{|z|=3} f(z)\,dz = 2\pi i\,[\operatorname{Res}(f;1) + \operatorname{Res}(f;-2)]$$

$z=-2$: By Prop 16.2 $f(z) = \frac{\varphi(z)}{(z+2)^2}$ where $\varphi(z) = \frac{z+1}{z-1}$ is analytic at a neighborhood of $z=-2$ and at $z=-2$, and $\varphi(-2) = \frac{1}{3} \ne 0 \Rightarrow$ (for $m=2$)

$$\operatorname{Res}(f;-2) = \frac{\varphi'(-2)}{1!} = \left(\frac{z+1}{z-1}\right)'\Big|_{z=-2} = -\frac{2}{(z-1)^2}\Big|_{z=-2} = -\frac{2}{9}$$

$z=1$: By Prop 16.2 $f(z) = \frac{\varphi(z)}{z-1}$, where $\varphi(z) = \frac{z+1}{(z+2)^2}$ is analytic at a neighborhood of $z=1$ (and at $z=1$), and $\varphi(1) = \frac{2}{9} \ne 0$

$\Rightarrow \operatorname{Res}(f;1) = \frac{\varphi(1)}{0!} = \frac{2}{9}$

$\Rightarrow \int_{|z|=3} f(z)\,dz = 2\pi i\left(\frac{2}{9} - \frac{2}{9}\right) = 0.$

Now we state (and prove some of) a collection of surprising and powerful consequences of Cauchy's Theorem.

Thm 20.1 (Gauss's Mean Value Theorem) Let f be analytic on and inside a circle of radius R, centered at z_0. Then, $f(z_0)$ is the mean value of f on C, that is

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta.$$

Pf: Parametrize the contour C by: $\gamma(\theta) = z_0 + Re^{i\theta}$, $0 \le \theta \le 2\pi$.
By the CIF and the definition of the contour integral (Def 17.7)

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{Re^{i\theta}}\, iRe^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta$$ ∎

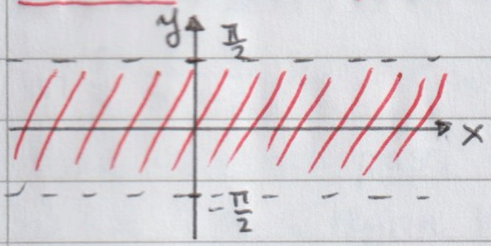
From this theorem follows that either $|f(z)| > |f(z_0)|$ for some z on C or $|f(z)| = |f(z_0)|$ for all z on C, in which case we can show that f must be a constant function [EXERCISE]

One can deduce:

Thm 20.2 (Maximum Modulus Principle) Let Ω be a bounded region. Let f be a continuous function on Ω and on the boundary of Ω - $\partial\Omega$. Let f be analytic on Ω. Then, either f is constant or else the maximum of $|f(z)|$ can only be attained on the boundary $\partial\Omega$ of Ω. That is, there exists $z_0 \in \partial\Omega$ such that $|f(z_0)| = M = \sup\{|f(z)| : z \in \Omega \cup \partial\Omega\}$ and $|f(z)| < M$ for all $z \in \Omega$.

Rmk: The Maximum Principal is false for unbounded region.

Ex 22.1: Let $\Omega = \{x+iy : -\frac{\pi}{2} < y < \frac{\pi}{2}\}$, let $f(z) = e^{e^z}$



Then, for $x \in \mathbb{R}$ we have $f(x \pm \frac{\pi}{2}i) = e^{\pm i e^x} \Rightarrow |f(z)| = 1$ for $z \in \partial\Omega$. However: $f(x) = e^{e^x} \xrightarrow[x\to\infty]{} \infty$ on the real axis ($y=0$) that is contained in Ω.

Ex 22.2: Obtain an estimate on the absolute value of the n-th derivative of an analytic function f in the disc $|z-z_0| \leq R$

Sol: By the Maximum Modulus Principle $\max|f(z)|$ is attained on the circle $|z-z_0| = R$ (depends only on R). Denote $\max|f(z)| = M_R$. To estimate $|f^{(n)}(z_0)|$ we use Cor 18.9 (extended CIF) and Prop 17.5 (ML inequality):

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n! M_R}{R^n}$$