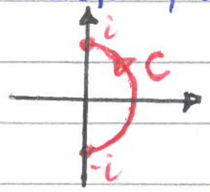
Problem Set 9 - Solutions

1) a) $C = \{z : |z| = 1, -\frac{\pi}{2} \le \arg z \le \frac{\pi}{2}\}$. Let $z = re^{i\theta}$. Since $|z| = 1 \Rightarrow r = 1$, since $-\frac{\pi}{2} \le \arg z \le \frac{\pi}{2}$ we get $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
Therefore, the contour is the right half of the unit circle.



The segment $[-i, i]$: $\arg z = \frac{\pi}{2}$ on the straight segment connecting 0 (not including) up to (including) i and $\arg z = -\frac{\pi}{2}$ on the lower part, however $|z| = 1$ only at $i, -i \Rightarrow$ this line segment is not a part of this contour. We parametrize: $\gamma(t) = e^{it}, -\frac{\pi}{2} \le t \le \frac{\pi}{2}$, $\gamma'(t) = ie^{it}dt$, therefore

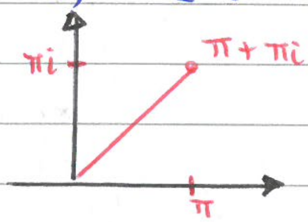
$$\int_C (z^3 + z\bar{z})dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{3it} + e^{it}e^{-it})ie^{it}\,dt = i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (e^{4it} + e^{it})\,dt$$

$$= i\left[\frac{1}{4i}e^{4it} + \frac{1}{i}e^{it}\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(\frac{1}{4}e^{it4} + e^{it}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{4}(e^{2\pi i} - e^{-2\pi i}) + (e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}) = 2i$$

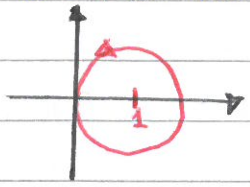
b) C is the straight line starting at $z_0 = 0$ and ending at $z_1 = \pi(1+i) \Rightarrow$ a parametrization that we can take is $\gamma(t) = t + it$, $0 \le t \le \pi \Rightarrow \gamma'(t) = (1+i)dt$ and we have

$$\int_C e^{2z}dz = \int_0^{\pi} e^{2t(1+i)}(1+i)dt = \frac{1}{2(1+i)}(1+i)e^{2t(1+i)}\Big|_0^{\pi} =$$
$$= \frac{1}{2}(e^{2\pi(1+i)} - 1) = \frac{1}{2}(e^{2\pi}e^{2\pi i} - 1) = \frac{1}{2}(e^{2\pi} - 1) \quad [e^{2\pi i} = 1!]$$

The contour:



c) $C = \{z : |z-1| = 1\}$ - the circle centered at $z = 1$ of radius 1



Parametrization: $\gamma(t) = 1 + e^{it}$, $0 \le t \le 2\pi$, $\gamma'(t) = ie^{it}dt$.
Therefore, we get:

$$\int_C (z^2 - 1)dz = \int_0^{2\pi} ((1+e^{it})^2 - 1)ie^{it}\,dt = i\int_0^{2\pi}(1 + e^{2it} + 2e^{it} - 1)e^{it}dt$$
$$= i\int_0^{2\pi}(e^{3it} + 2e^{2it})\,dt = i\left(\frac{1}{3i}e^{3it} + \frac{2}{2i}e^{2it}\right)\Big|_0^{2\pi} =$$
$$= \left(\frac{1}{3}e^{3it} + e^{2it}\right)\Big|_0^{2\pi} = \frac{1}{3}(e^{6i\pi} - 1) + (e^{4i\pi} - 1) = 0 \quad (e^{2\pi ki} = 1 \ \forall k \in \mathbb{Z})$$

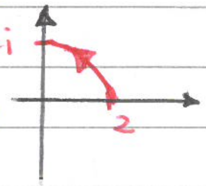
d) We use the same parametrization as in c):
$\gamma(t) = 1 + e^{it}$, $0 \le t \le 2\pi$, $\gamma'(t) = ie^{it}dt$

Note that:

$x-y+i(x+y) = (x+iy)+(ix-y) = (x+iy)+i(x-\frac{1}{i}y) = (x+iy)+$
$+i(x+iy) = z+iz = z(1+i)$

$\Rightarrow \int_C [(x-y)+i(x+y)]\,dz = \int_C z(1+i)\,dz = (1+i)\int_C z\,dz =$
$= (1+i)\int_0^{2\pi}(1+e^{it})ie^{it}\,dt = i(1+i)\int_0^{2\pi}(e^{it}+e^{2it})\,dt =$
$= i(1+i)\left(\frac{1}{i}e^{it}+\frac{1}{2i}e^{2it}\right)\Big|_0^{2\pi} = (1+i)\left(e^{it}+\frac{1}{2}e^{2it}\right)\Big|_0^{2\pi} =$
$= (1+i)\left((e^{2\pi i}-1)+\frac{1}{2}(e^{4\pi i}-1)\right) = 0.$

2) a) Let us draw the contour



By Prop 1.2 for any $z, w \in \mathbb{C}$ we have $|z-w| \ge ||z|-|w||$, therefore:

(*) $|z^2-1| \ge ||z|^2-1| = |4-1| = 3$ (since $|z|=2$)

By Prop 17.2: $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$.

Parametrize the contour by: $\gamma(t) = 2e^{it}$, $0 \le t \le \frac{\pi}{2}$, then on this contour, by Prop 17.2

$$\left|\int_C \frac{dz}{z^2-1}\right| \le \int_C \frac{|dz|}{|z^2-1|} = \int_0^{\frac{\pi}{2}} \frac{|2ie^{it}|}{|4e^{2it}-1|}\,dt \underset{\text{by (*) and } |e^{it}|=1\ \forall t}{\le} \int_0^{\frac{\pi}{2}} \frac{1}{3}\,2\,dt =$$

$= \frac{2}{3}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{3}$ $\{|e^{it}| = |\cos t + i\sin t| = \sqrt{\cos^2 t + \sin^2 t} = 1\ \forall t\}$.

b) The ML Inequality: $\left|\int_C f(z)\,dz\right| \le ML$, where $|f(z)| \le M$ for any $z \in C$ and $L = \text{length}(C)$.

Let us estimate $\int_C \frac{e^{2z}}{6z^5}\,dz$ for C: the unit circle traversed once anticlockwise.

Note: for $|z|=1$ (namely, $z \in C$):

$$|f(z)| = \left|\frac{e^{2z}}{6z^5}\right| = \frac{|e^{2x}e^{2iy}|}{6|z|^5} = \frac{|e^{2x}||e^{2iy}|}{6|z|^5} \underset{|e^{2iy}|=1,\ |z|=1}{=} \frac{|e^{2x}|}{6}$$

$\Rightarrow$ since for $|z|=1$ $x \le 1$: $|f(z)| \le \frac{e^2}{6} = M$

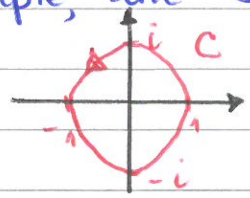
$\text{length}(C) = 2\pi = L \Rightarrow \left|\int_C \frac{e^{2z}}{6z^5}\,dz\right| \le \frac{e^2}{6}\,2\pi = \frac{\pi e^2}{3}.$

3) a) Cauchy's Theorem (Thm 18.3): If C is any simple, closed contour in $\mathbb{C}$ and f is analytic everywhere on and inside C, then $\int_C f(z)\,dz = 0$.

$C = \{z \in \mathbb{C} : |z|=1\}$ - it is the unit circle centered at $z=0$.

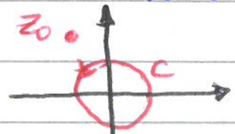
This contour is closed and simple: parametrized by $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$ and for any $t_1 \ne t_2$ $(\ne 0, 2\pi)$ $\gamma(t_1) \ne \gamma(t_2) \Rightarrow$

simple, and $\gamma(0) = \gamma(2\pi) = 1 \Rightarrow$ closed.



To prove the statement we need to check that all the functions in i) → v) are analytic everywhere on and inside C $\Rightarrow$ by Cauchy's Theorem $\int_C f(z)dz = 0$.

i) $f(z) = \dfrac{z^3}{z+1-2i}$: f is a rational function $\Rightarrow$ (by Ex 9.2) it is analytic everywhere except for the roots of the denominator.



$z+1-2i = 0 \iff z = -1+2i \ (= z_0)$

Since z_0 is not inside and not on the contour C, $f(z)$ is analytic everywhere on and inside C $\Rightarrow$ by Cauchy's Theorem $\int_C f(z)dz = 0$.

ii) $f(z) = ze^{-z}$: this is an entire function, since by Ex 9.1 $f_1(z) = z$ is an entire function, by Prop 12.3 $f_2(z) = e^{-z}$ is an entire function $\Rightarrow$ by Prop 7.1 $f(z) = f_1(z)f_2(z) = ze^{-z}$ is an entire function (analytic for every $z \in \mathbb{C}$), in particular $f(z)$ is analytic on and inside C. Thus, by Cauchy's Theorem $\int_C f(z)dz = 0$.

iii) $f(z) = \dfrac{1}{z^2+3z+4}$: as in i) f is analytic everywhere except at the roots of the denominator:

$$z^2+3z+4 = 0 \iff z_{1,2} = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

Since $|z_{1,2}| = \left|-\frac{3}{2} + i\frac{\sqrt{7}}{2}\right| = 2 \notin C$ (and it is outside of C) $\Rightarrow$ $f(z)$ is analytic everywhere on and inside C $\Rightarrow$ by Cauchy's Theorem $\int_C f(z)dz = 0$.

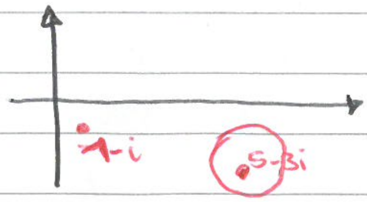
iv) $f(z) = \mathrm{Ln}(z-4)$: The branch point of $f(z)$ $z = 4$ (f is not analytic at that point and on any small neighborhood of that point) $\Rightarrow$ f is analytic on and inside C, hence, by Cauchy's Theorem $\int_C f(z)dz = 0$

v) $f(z) = \cos z$. From Def 12.5 $f(z)$ is an entire function (Check!), therefore, in particular, it is analytic everywhere on and inside C $\Rightarrow$ by Cauchy's Theorem $\int_C f(z)dz = 0$.

b) $f(z) = \dfrac{z^2}{z-1+i}$ - not analytic at $z_0 = 1-i$

i) $C_1 = \{z \in \mathbb{C} : |z+3i-5| = 1\}$ - it is a circle centered at $z_0 = 5-3i$

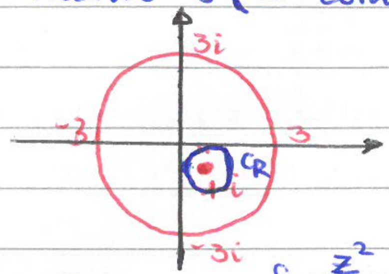
of radius 1.



The only singularity of $f(z)$ is the root of the denominator $z_0 = 1-i$.

$\Rightarrow$ f is analytic everywhere on and inside C_1 $\Rightarrow$ by Cauchy's Theorem $\int_{C_1} f(z)\,dz = 0$.

ii) $C_2 = \{z \in \mathbb{C} : |z| = 3\}$ - this is a circle centered at the origin of radius 3 ($\Rightarrow$ contains the singularity of f $z_0 = 1-i$)



To compute the integral we use the Deformation Principle (Cor 18.4) Let C_R be the circular contour centered at $1-i$ of radius 1. By the Deformation Principle: $\int_C \frac{z^2}{z-1+i}\,dz = \int_{C_R} \frac{z^2}{z-1+i}\,dz$, since C and C_R are both simple, closed, positively oriented contours, C_R lies interior to C, and f is analytic in a domain that contains both C and C_R, and the region between them {the only singularity of f is a simple pole at $z = 1-i$}. Parametrize C_R by:

$\gamma(t) = 1-i+e^{it}$, $0 \le t \le 2\pi$ {in general, for a circle centered at z_0 of radius R $\gamma(t) = z_0 + Re^{it}$, $0 \le t \le 2\pi$. Here: $z_0 = 1-i$, $R = 1$}, then $\gamma'(t)\,dt = ie^{it}\,dt$, and we have:

$$\int_C \frac{z^2}{z-1+i}\,dz = \int_{C_R} \frac{z^2}{z-1+i}\,dz = \int_0^{2\pi} \frac{(1-i+e^{it})^2}{1-i+e^{it}-1+i}\, ie^{it}\,dt =$$

$$= i\int_0^{2\pi} \left[(1-i)^2 + e^{2it} + 2e^{it}(1-i)\right] dt = i\int_0^{2\pi} \left[-2i + e^{2it} + 2e^{it}(1-i)\right] dt$$

$$= i\left[-2it + \frac{1}{2i}e^{2it} + \frac{2(1-i)}{i}e^{it}\right]\Big|_0^{2\pi} = \left[2t + \frac{1}{2}e^{2it} + 2(1-i)e^{it}\right]\Big|_0^{2\pi}$$

$$= 2\cdot 2\pi + \frac{1}{2}\left(e^{4\pi i}-1\right) + 2(1-i)\left(e^{2\pi i}-1\right) = 4\pi.$$

4) a) $\int_{|z|=5} \frac{dz}{z^2+5z+4}$: Let us first find the roots of the denominator, that is the singularities of $f(z) = \frac{1}{z^2+5z+4}$.

$z^2+5z+4 = 0 \iff z_{1,2} = \begin{matrix}-1\\-4\end{matrix} \Rightarrow z^2+5z+4 = (z+1)(z+4)$,

$z=-1$ and $z=-4$ are simple poles, both inside $\{z \in \mathbb{C} : |z| = 5\}$.

Note: $\frac{1}{z^2+5z+4} = \frac{1}{3}\left[\frac{1}{z+1} - \frac{1}{z+4}\right]$.

To compute the integral we use the generalization of the Deformation Principle (Cor 18.6) as follows.

Let C_1 be a circular contour centered at $z=-1$ of radius 1, and let C_2 be a circular contour centered at $z=-4$ of radius $\frac{1}{2}$ (so it will lie interior to C!) Then, f is analytic in a domain containing $C=\{z\in\mathbb{C}, |z|=5\}$, C_1, C_2 and the region between them, thus by Cor 18.6

$$\int_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

$$\int_{C_1} f(z)\,dz = \int_{\substack{|z+1|=1 \\ |z+4|=\frac{1}{2}}} \frac{1}{3}\left[\frac{1}{z+1} - \frac{1}{z+4}\right]dz = \frac{1}{3}\int_{|z+1|=1}\frac{dz}{z+1} - \frac{1}{3}\int_{|z+4|=1/2}\frac{dz}{z+4}$$

Note: $\tilde{f}(z) = \frac{1}{z+4}$ is analytic on and inside C_1, therefore, by Cauchy's Theorem $\int_{C_1}\tilde{f}(z)\,dz = 0$

$$\Rightarrow \int_{C_1} f(z)\,dz = \frac{1}{3}\int_{|z+1|=1}\frac{dz}{z+1} = \frac{1}{3}\int_0^{2\pi}\frac{1}{-1+e^{it}+1}ie^{it}\,dt = \frac{1}{3}\int_0^{2\pi} i\,dt = \frac{2\pi i}{3}.$$

$\gamma(t) = -1+e^{it}$, $0\le t\le 2\pi$

In the same way we conclude that $\int_{C_2}\frac{dz}{z+4}=0$, thus

$$\int_{C_2} f(z)\,dz = -\frac{1}{3}\int_{C_2}\frac{dz}{z+4} = -\frac{1}{3}\int_0^{2\pi}\frac{1}{-4+\frac{1}{2}e^{it}+4}\cdot\frac{i}{2}e^{it}\,dt = -\frac{2\pi i}{3}$$

$\gamma(t) = -4+\frac{1}{2}e^{it}$, $0\le t\le 2\pi$

$$\Rightarrow \int_C f(z)\,dz = \frac{1}{3}\left[\int_{|z+1|=1}\frac{dz}{z+1} - \int_{|z+4|=1/2}\frac{dz}{z+4}\right] = \frac{2\pi i}{3} - \frac{2\pi i}{3} = 0.$$

b) $f(z) = \frac{\sin z}{z^3+16z} = \frac{\sin z}{z(z^2+16)} = \frac{\sin z}{z}\cdot\frac{1}{(z-4i)(z+4i)}$

$\Rightarrow$ f has a removable singularity at $z=0$ and 2 simple poles at $z=\pm 4i$ (Justify!)

The contour $\{z\in\mathbb{C} : |z+2i|=1\}$ is a circle centered at $z=-2i$ of radius 1 $\overset{(\text{why?})}{\Rightarrow}$ $0, 4i, -4i \notin \{z\in\mathbb{C} : |z+2i|=1\}$ $\Rightarrow$ on and inside this contour $f(z)$ is analytic. C is a simple, closed, positively oriented contour $\Rightarrow$ by Cauchy's Theorem

$$\int_C f(z)\,dz = \int_C \frac{\sin z}{z^3+16z}\,dz = 0.$$