Week 10. Lecture 25

We have started to study the complex integration, formulated and proved Prop 17.3 about contour integrals. We conclude the following useful corollary:

Cor 17.4: Let f be continuous on a domain D, with antiderivative F on D.

1) If C_1 and C_2 are any two contours both starting at z_0 and both ending at z_1, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

2) If C is a closed contour in D, then $\int_C f(z)\,dz = 0$

Pf: 1) Immediate from Prop 17.3 4)

2) If C is a closed contour, then we can write $C = C_1 - C_2$, where C_1 and C_2 have the same start point and the same end point. Then

$$\int_C f(z)\,dz = \int_{C_1-C_2} f(z)\,dz \underset{\text{Prop 17.3 1)}}{=} \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0 \quad \blacksquare$$

Note: The condition that f has antiderivative is necessary!

Ex 19.11: Compute $\int_C f(z)\,dz$ where $f(z) = \bar{z}$ and C connects 2 points $z_0 = 0$, $z_1 = 2+4i$

1) by straight line segment 2) by parabola (part of it) $y = x^2$

Sol: 1) Straight line connecting 0 and $2+4i$ is parametrized by $\gamma(t) = (2+4i)t$, $0 \le t \le 1$. Thus,

$$\int_C f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 \overline{(2+4i)t}\,(2+4i)\,dt = \int_0^1 20t\,dt = 10$$

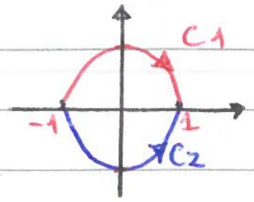
2) Parametrization: $\gamma(t) = t + it^2$, $0 \le t \le 2$, and

$$\int_C f(z)\,dz = \int_0^2 f(\gamma(t))\gamma'(t)\,dt = \int_0^2 \overline{(t+it^2)}(1+2it)\,dt =$$
$$= \int_0^2 (t-it^2)(1+2it)\,dt = \int_0^2 (t+it^2+2t^3)\,dt = 10 + i\tfrac{8}{3} \; (\neq 10)$$

$\bar{z}$ does not have an antiderivative!

Ex 19.12: Let C_1 be the contour from -1 to 1 along the upper unit semicircle and C_2 the contour from -1 to 1 along the lower unit semicircle. Compute $\int_{C_1} \frac{1}{(z-4)^2}\,dz$

Sol:



Note: $\int_{C_1 - C_2} \frac{1}{(z-4)^2}\,dz = 0$ This is an analytic function on the unit circle, the only singularity is a pole of order 2 at $z = 4$ - outside the unit circle, with antiderivative $-\frac{1}{z-4}$.

$\Rightarrow$ By Cor 17.4 2): $\int_{C_1 - C_2} \frac{1}{(z-4)^2}\,dz = 0$ $\{C_1 - C_2$ is a closed contour, the unit circle$\}$

$\Rightarrow \int_{C_1} \frac{dz}{(z-4)^2} = \int_{C_2} \frac{dz}{(z-4)^2} \Rightarrow \int_{C_1} \frac{dz}{(z-4)^2} = -\frac{1}{z-4}\Big|_{-1}^{1} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

Ex 19.13: Compute $\int_C \frac{dz}{(z-z_0)^n}$, where $C = \{z \in \mathbb{C} : |z - z_0| = R\}$ and n is some integer.

Sol: C is a circle centered at z_0 of radius R. Parametrization of C: $|z - z_0| = R \Rightarrow (x-x_0)^2 + (y-y_0)^2 = R^2$. Take $x - x_0 = R\cos t$, $y - y_0 = R\sin t$, $0 \le t \le 2\pi$, then by Euler's formula

$$z - z_0 = R(\cos t + i\sin t) = Re^{it}, \quad 0 \le t \le 2\pi \Rightarrow dz = iRe^{it}\,dt$$

By Def 17.7 (of the contour integral)

$$\int_C \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{iRe^{it}}{R^n e^{int}}\,dt = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{-i(n-1)t}\,dt$$

Consider two cases:

$n = 1$: $\int_C \frac{dz}{z - z_0} = i\int_0^{2\pi} dt = 2\pi i$

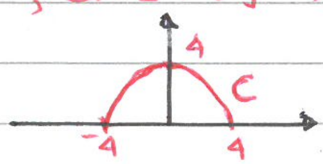
$n \neq 1$: $\int_C \frac{dz}{(z-z_0)^n} = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{-i(n-1)t}\,dt = \frac{i}{i(1-n)R^{n-1}} e^{i(1-n)t}\Big|_0^{2\pi}$

$$= \frac{1}{(1-n)R^{n-1}}\left[e^{i(1-n)2\pi} - 1\right] = 0 \quad \{e^{2\pi k i} = 1 \ \forall k \in \mathbb{Z}\}$$

Ex 19.4: Compute $\int_C \frac{dz}{\sqrt{z}}$ where $C = \{(x,y) : x^2 + y^2 = 16, y > 0\}$, and the branch of $\sqrt{z}$ is such that $\sqrt{4} = -2$.

Sol: Note: the contour is the upper half of the circle centered at 0 of radius 4.

Rewrite: $C = \{z \in \mathbb{C} : |z| = 4, \operatorname{Im} z > 0\} \Rightarrow$ if $z \in C$, then $z = 4e^{it}$ for $0 \le t \le \pi$



$\sqrt{z}$ attains 2 values:

$$w_1 = \sqrt{4e^{it}} = 2e^{i\frac{t}{2}} \quad \text{and} \quad w_2 = \sqrt{4e^{it}} = 2e^{i(\frac{t}{2} + \pi)}$$

If $t = 0$: $w_1 = \sqrt{4} = 2e^0 = 2$ and $w_2 = \sqrt{4} = 2e^{i\pi} = -2 \Rightarrow$ the given branch is w_2. Now we compute:

$\sqrt{z} = 2e^{i(\frac{t}{2} + \pi)}$ $\quad dz = 4ie^{it}\,dt$

$$\Rightarrow \int_C \frac{dz}{\sqrt{z}} = \int_0^{\pi} \frac{4ie^{it}}{2e^{i(\frac{t}{2}+\pi)}}\,dt \underset{\text{Prop 17.1 3)}}{=} 2i\int_0^{\pi} e^{i(\frac{t}{2} - \pi)}\,dt = 4e^{i(\frac{t}{2} - \pi)}\Big|_0^{\pi}$$

$= 4(e^{-i\frac{\pi}{2}} - e^{-i\pi}) = 4(-i + 1)$ $\quad \{e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i\}$

The estimate in Prop 17.2 for the size of an integral of a complex-valued function of a real variable also has an important consequence for contour integrals.

Prop 17.5 (ML inequality): Let C be a contour. If $|f(z)| \le M$ for all $z \in C$ and Length $(C) = L$, then $\left| \int_C f(z)\,dz \right| \le ML$

Pf: $\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\,dt \right| \underset{\text{Prop 17.2}}{\le} \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$

$\le M \int_a^b |\gamma'(t)|\,dt \underset{\text{Def 17.5}}{=} ML$ ☒

Now that we have defined contour integrals and discussed some of their properties, we concentrate on one of the key theorems of complex analysis - Cauchy's Theorem.

Cauchy's Theorem

Def 18.1: $U \subseteq \mathbb{C}$ is convex if for all $z_1, z_2 \in U$, the line segment $[z_1, z_2] := \{ t z_2 + (1-t) z_1 : 0 \le t \le 1 \} \subseteq U$ (is contained in U).
$U \subseteq \mathbb{C}$ is star-shaped about w if for all $z \in U$, the line segment $[z, w] \subseteq U$.

Rmk: Any convex set is star-shaped, but not every star-shaped set is convex: Look at  - it is star-shaped about w (in the center), but it is not convex - the line segment connecting any two neighboring corners is not contained in this set.

Rmk: Set is convex if and only if it is star-shaped with respect to any point in that set. Note: star-shaped set is always path-connected, thus it is always connected.

Our goal now is to prove Cauchy's Theorem for star-shaped domains.

Thm 18.1 (Cauchy's Thm for star-shaped domains)
Let f be a function which is analytic on an open star-shaped region $U \subseteq \mathbb{C}$. Then, for every closed contour C in U

$$\int_C f(z)\,dz = 0.$$

Pf: It will suffice to show that f has an antiderivative F on U, since then by Cor 17.4 2) we will conclude for a closed contour C that $\int_C f(z)\,dz = 0$.

Define: $F(z) = \int_{[w,z]} f(\zeta)\,d\zeta$, where w is such that U is star-shaped about w, and $[w,z]$ is the straight line segment from w to z.

Claim: F is differentiable and $F'(z) = f(z)$

Pf of Claim: Consider $\dfrac{F(z)-F(z_0)}{z-z_0} = \dfrac{1}{z-z_0}\left[\int_{[w,z]} f(\zeta)\,d\zeta - \int_{[w,z_0]} f(\zeta)\,d\zeta\right]$

Suppose we could show that

$$(*) \quad \int_{[w,z]} f(\zeta)\,d\zeta - \int_{[w,z_0]} f(\zeta)\,d\zeta = \int_{[z,z_0]} f(\zeta)\,d\zeta$$

Then, by $(*)$: $\dfrac{F(z)-F(z_0)}{z-z_0} = \dfrac{1}{z-z_0}\int_{[z,z_0]} f(\zeta)\,d\zeta$

f is analytic on $U \Rightarrow$ by Prop 6.5 f is continuous on $U \Rightarrow$ for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $|\zeta - z_0| < \delta$, then $|f(\zeta) - f(z_0)| < \varepsilon$
$\Rightarrow$ If $|\zeta - z_0| < \delta$, then

$\left|\int_{[z_0,z]} f(\zeta)\,d\zeta - f(z_0)(z-z_0)\right| < \varepsilon|z-z_0|$, since

$\left|\int_{[z_0,z]} f(\zeta)\,d\zeta - \int_{[z_0,z]} f(z_0)\,d\zeta\right| < \varepsilon|z-z_0|$

Thus: $\left|\dfrac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| < \varepsilon \Rightarrow F$ is differentiable at z_0, with derivative $F'(z_0) = f(z_0) \Rightarrow$ we proved Cauchy's Theorem up to $(*)$. It remains to prove $(*)$ which is Cauchy's Theorem for a Triangle. ▨

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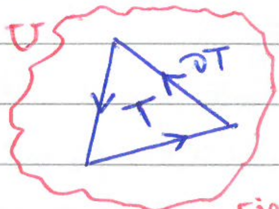
We have proved Cauchy's Theorem for star-shaped domains up to: ⊛ $\int_{[w,z]} f(z)dz - \int_{[w,z_0]} f(z)dz = \int_{[z,z_0]} f(z)dz$,

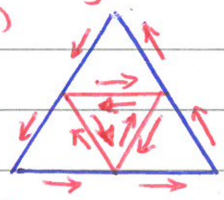
which is Cauchy's Theorem for a Triangle. Now we prove it.

Thm 18.2 (Cauchy's Theorem for a triangle): Let f be analytic in a domain U containing a triangle T. Then: $\int_{\partial T} f(z)dz = 0$.

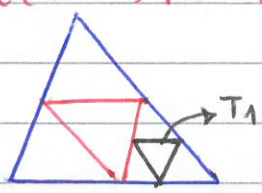
Pf: Let $\eta(T) = \int_{\partial T} f(z)dz$, denote the length of ∂T by L.



Divide T into 4 triangles $T^{(j)}$ by bisecting the sides of T:



By orienting each of the boundaries $\partial T^{(j)}$ of the subtriangles anticlockwise, we have: $\eta(T) = \sum_{j=1}^{4} \eta(T^{(j)})$ {the internal edges of each of the subtriangles $T^{(j)}$ will cancel out} Each $T^{(j)}$ has a boundary with Length $(\partial T^{(j)}) = \frac{L}{2}$. At least one of the $T^{(j)}$ must satisfy $|\eta(T^{(j)})| \geq \frac{1}{4}|\eta(T)|$. Call this triangle T_1.



Divide T_1 into 4 triangles in the same way as T and conclude that one of these triangles, T_2, satisfies $|\eta(T_2)| \geq \frac{1}{4}|\eta(T_1)| \geq \frac{1}{4^2}|\eta(T)|$,

Length $(\partial T_2) = \frac{L}{4}$. Repeating this procedure, we construct a sequence of triangles $T_1 \supset T_2 \supset T_3 \supset \dots \supset T_n \supset \dots$ with $|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T)|$ and Length $(\partial T_n) = \frac{L}{2^n}$. Let z_0 be the point of intersection $\bigcap_{n=1}^{\infty} T_n$ {exists and unique by Cantor Lemma or ⊛⊛} Since f is differentiable at z_0, given any $\varepsilon > 0$ there exists $\delta > 0$ such that for $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0|$$

Take n such that $\frac{L}{2^n} < \delta$. Then $\varepsilon|z - z_0| < \frac{L\varepsilon}{2^n}$ for every $z \in \partial T_n$, so $\left|\int_{\partial T_n} [f(z) - f(z_0) - f'(z_0)(z-z_0)]dz\right| \leq \frac{L\varepsilon}{2^n} \cdot \frac{L}{2^n} = \frac{L^2}{4^n}\varepsilon$

However:

$$\int_{\partial T_n} [f(z_0) + f'(z_0)(z - z_0)]dz = 0,$$

since $f(z_0) + f'(z_0)(z - z_0)$ has antiderivative $z f(z_0) + f'(z_0)\left(\frac{z^2}{2} - z z_0\right)$ and ∂T_n is a closed curve (Cor 17.4) Hence:

$$|\eta(T_n)| = \left|\int_{\partial T_n} f(z)dz\right| \leq (L^2/4^n)\varepsilon$$

On the other hand we know that $|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T)|$
$\Rightarrow |\eta(T)| \leq \varepsilon L^2$. Since $\varepsilon > 0$ is arbitrary $\eta(T) = 0$ ▨

(**) Existence of z_0: On each step of the process choose $z_n \in T_n$ (a point) and obtain an infinite sequence $\{z_n\}_{n=1}^{\infty}$. Then:

1) $\{z_n\}$ is a Cauchy sequence, namely, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any pair $n, m > N$ $|z_n - z_m| < \varepsilon$.
First, it is bounded: Set $\varepsilon = 1$ and choose N such that for any $n, m \in N$ $|z_n - z_m| < 1$ (using that $\{z_n\}$ is a cauchy sequence). Namely:
(*) $||z_n| - |z_m|| < |z_n - z_m| < 1$. Fix $m_0 \in \mathbb{N}$. From (*) for all $n > N$
$|z_{m_0}| - 1 < |z_n| < |z_{m_0}| + 1 \Rightarrow$ for all $n \geq 1$
$\min\{|z_1|, \dots, |z_{m_0-1}|, |z_{m_0}| - 1\} \leq |z_n| \leq \max\{|z_1|, \dots, |z_{m_0-1}|, |z_{m_0}| + 1$
$\Rightarrow$ it is bounded. Now apply to $\{z_n\}$ Bolzano-Weierstrass Thm and obtain a converging subsequence $\{z_{n_k}\}$, $\lim_{k\to\infty} z_{n_k} = z_0$ (apply separately to $\mathrm{Re}\, z_n$ and $\mathrm{Im}\, z_n$)

2) $\lim_{n\to\infty} z_n = z_0$: Fix $\varepsilon > 0$, choose N_1 s.t. $\forall n, m > N_1$ $|z_n - z_m| < \frac{\varepsilon}{2}$ (cauchy property). Choose N_2 so that $\forall n_k > N_2$ and such that z_{n_k} belongs to the subsequence we have $|z_0 - z_{n_k}| < \frac{\varepsilon}{2}$ (use that $\lim_{k\to\infty} z_{n_k} = z_0$). Set $N = \max(N_1, N_2)$ and fix $n_k > N$ such that z_{n_k} belongs to the subsequence. Then, for any $n > N$

$$|z_0 - z_n| = |z_0 - z_{n_k} + z_{n_k} - z_n| \leq |z_{n_k} - z_0| + |z_n - z_{n_k}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow \lim_{n\to\infty} z_n = z_0$.

3) Let us show that $\{z_n\}$ is a Cauchy sequence: $z_1 \in T_1$, $z_2 \in T_2, \dots$
Since the diameter of each triangle shrinks by a factor 2 (at each generation) and since $\mathrm{diam}(T_n) \leq \mathrm{Length}(\partial T_n)$ we get for sufficiently large n: $|z_n - z_m| < \mathrm{Length}(\partial T_n) < \frac{L}{2^n} < \varepsilon \Rightarrow \{z_n\}$ is a Cauchy sequence that converges to z_0 ▨

From the statement of Cauchy's Theorem for a star-shaped region, one can deduce the same result for more general regions comprised of star-shaped pieces.

From Thm 18.1 follows: If a curve C can be expressed as $C = C_1 + C_2 + \dots + C_N$ where the region enclosed by each C_j is star-shaped, then

$$\oint f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \dots + \int_{C_N} f(z)dz = 0.$$

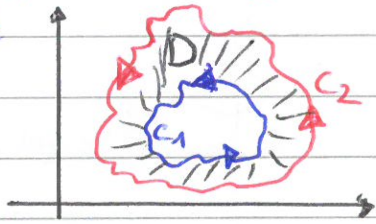
As a consequence one can prove:

Thm 18.3 (Cauchy's Theorem): If C is any simple, closed contour in $\mathbb{C}$ and f is analytic everywhere on and inside C, then $\int_C f(z)dz = 0$.

One of very useful corollaries of this theorem is the Deformation Principle.

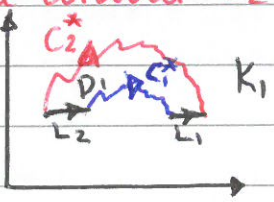
Cor 18.4 (Deformation Principle) Let C_1 and C_2 be two simple, closed, positively oriented contours such that C_1 lies interior to C_2. If f is analytic in a domain D that contains both C_1 and C_2 and the region between them, then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.

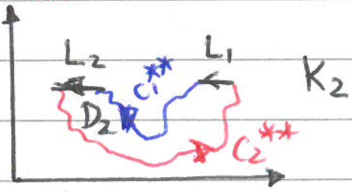
The picture that you should have in mind here is where D is a domain that contains both curves and the region between them:



Pf: The main idea of the proof is to divide the region between C_1 and C_2 by "cuts" into star-shaped regions.

Construct 2 disjoint contours (or cuts) L_1 and L_2 that join C_1 and C_2. The contour C_1 is cut into 2 contours C_1^* and C_1^{**} and the contour C_2 is cut into C_2^* and C_2^{**}.





Now form 2 new contours:

$K_1 = -C_1^* + L_1 + C_2^* - L_2$ {$-L_2, -C_1^*$ since we have changed the orientation}

$K_2 = -C_1^{**} + L_2 + C_2^{**} - L_1$

$K_{1,2}$ - new contours, $D_{1,2}$ - new domains.

Now we can apply Thm 18.3 (Cauchy's Thm) to the contours K_1 and K_2 and obtain: $\int_{K_1} f(z)dz = 0 \quad \int_{K_2} f(z)dz = 0$

Adding contours (to reconstruct the original one) gives us:

$K_1 + K_2 = -C_1^* + L_1 + C_2^* - L_2 - C_1^{**} + L_2 + C_2^{**} - L_1 = C_2^* + C_2^{**} - C_1^* - C_1^{**}$

$= C_2^* + C_2^{**} - (C_1^* + C_1^{**}) = C_2 - C_1$

$$\Rightarrow \quad 0 = \int_{K_1} f(z)\,dz + \int_{K_2} f(z)\,dz \overset{\text{Prop 17.3 1)}}{=} \int_{K_1+K_2} f(z)\,dz = \int_{C_2-C_1} f(z)\,dz = \int_{C_2} f(z)\,dz - \int_{C_1} f(z)\,dz$$

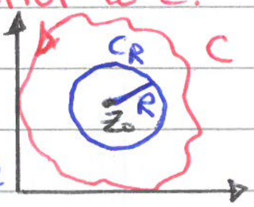
Namely: $\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz$. ▨

Now we can give an alternative proof to Ex 19.13. It is an important tool for computations.

Cor 18.5: Let $z_0 \in \mathbb{C}$ be fixed. If C is a simple, closed, positively oriented contour such that z_0 lies interior to C, then

$$\int_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n \neq 1, n \in \mathbb{Z} \end{cases}$$

Pf: Since z_0 lies interior to C, we can choose R so that the circle C_R with center z_0 and radius R lies interior to C. Hence, $f(z) = \frac{1}{(z-z_0)^n}$ is analytic in a domain D that contains both C and C_R and the region between them {we excluded the only singularity of f $z=z_0$, everywhere else f is analytic}



As in Ex 19.13: Let C_R be parametrized by: $z(\theta) = z_0 + Re^{i\theta}$, $dz(\theta) = iRe^{i\theta}\,d\theta$, $0 \le \theta \le 2\pi$.

The deformation principle implies that $\int_C f(z)\,dz = \int_{C_R} f(z)\,dz$

$$\Rightarrow \int_C \frac{dz}{(z-z_0)^n} = \int_{C_R} \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{iRe^{i\theta}}{R^n e^{in\theta}}\,d\theta = \frac{i}{R^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta}\,d\theta$$

$$= \begin{cases} i\int_0^{2\pi} d\theta = 2\pi i & n=1 \\ \frac{1}{(1-n)R^{n-1}} e^{i(1-n)\theta}\Big|_0^{2\pi} = \frac{1}{(1-n)R^{n-1}} - \frac{1}{(1-n)R^{n-1}} = 0 & n \neq 1 \end{cases}$$

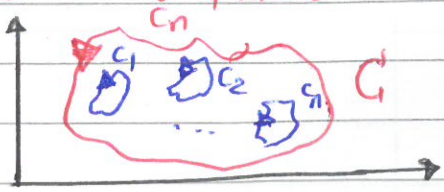
▨

More generally, a similar proof to the one used for the Deformation Principle, gives

Cor 18.6: If C is a simple, closed, positively oriented contour, and $C_1, C_2, .., C_n$ are simple, closed, positively oriented contours inside C with disjoint interiors, and f is analytic in the region between C and $C_1, C_2, .., C_n$, then

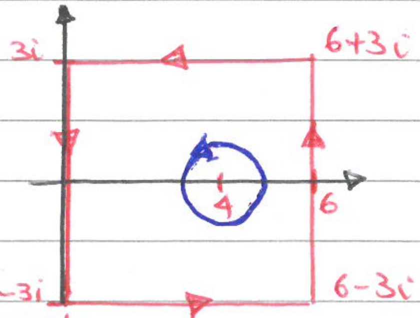
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + .. + \int_{C_n} f(z)\,dz.$$

The picture to have in mind is:



Ex 20.1: Compute $\int_C \frac{z}{(z-4)^2}dz$, where C is the contour defined by traversing once the square with vertices $\pm 3i$, $6 \pm 3i$ anticlockwise.

Sol: To compute this integral we apply the Deformation principle and observe that $\int_C \frac{z\,dz}{(z-4)^2} = \int_{C'} \frac{z\,dz}{(z-4)^2}$ where C' is the circular contour centered at 4 of radius 1. In particular, C' may be parametrized by the path $\gamma(t) = 4 + e^{2\pi i t}$, $0 \le t \le 1$



$$\Rightarrow \int_{C'} \frac{z\,dz}{(z-4)^2} = \int_0^1 \frac{4+e^{2\pi i t}}{e^{4\pi i t}} 2\pi i e^{2\pi i t}\,dt = 2\pi i \int_0^1 (4e^{-2\pi i t} + 1)\,dt$$

$$= 2\pi i \left[-\frac{2}{\pi i} e^{-2\pi i t} + t \right]\Big|_0^1 = \left[-4e^{-2\pi i t} + 2\pi i t \right]\Big|_0^1 = -4 + 2\pi i - (-4+0) = 2\pi i.$$