Problem Set 8 - Solutions.

1) a) 1) $f(z) = z^3 \cos z$: $f(z)$ has a zero at $z=0$ and has infinitely many zeros at $z_k = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$. Let us see what is the order of these zeros.

$z=0$: Rewrite $f(z)$ using the Taylor series expansion of $\cos z$ at 0:
$f(z) = z^3 \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) = z^3 \varphi(z)$, where $\varphi(z)$ is analytic at 0, $\varphi(0) = 1 \neq 0 \Rightarrow z=0$ is a zero of order $\underline{3}$.

$z_k = \frac{\pi}{2} + \pi k$: Rewrite $f(z)$ using the Taylor series expansion of $\cos z$ about $z_k = \frac{\pi}{2} + \pi k$:

First, we find the Taylor series expansion: $f(z) = \cos z$, $z_0 = \frac{\pi}{2} + \pi k$

$f(z_0) = 0$; $f'(z_0) = -\sin\left(\frac{\pi}{2} + \pi k\right) = -(-1)^k = (-1)^{k+1}$

$f''(z_0) = -\cos\left(\frac{\pi}{2} + \pi k\right) = 0$, $f'''(z_0) = \sin\left(\frac{\pi}{2} + \pi k\right) = (-1)^k, \dots$

$$\Rightarrow \cos z = (-1)^{k+1}\left(z - \left(\frac{\pi}{2} + \pi k\right)\right) + (-1)^k \frac{\left(z - \left(\frac{\pi}{2} + \pi k\right)\right)^3}{3!} + \dots$$

$$\Rightarrow z^3 \cos z = z^3 \left((-1)^{k+1}\left(z - \left(\frac{\pi}{2} + \pi k\right)\right) + (-1)^k \frac{\left(z - \left(\frac{\pi}{2} + \pi k\right)\right)^3}{3!} + \dots \right)$$

$$= \left(z - \left(\frac{\pi}{2} + \pi k\right)\right) z^3 \left((-1)^{k+1} + \frac{(-1)^k}{3!}\left(z - \left(\frac{\pi}{2} + \pi k\right)\right)^2 + \dots \right)$$

$$= \left(z - \left(\frac{\pi}{2} + \pi k\right)\right) \varphi(z),$$

where $\varphi(z) = z^3 \left((-1)^{k+1} + \frac{(-1)^k}{3!}\left(z - \left(\frac{\pi}{2} + \pi k\right)\right)^2 + \dots \right)$ is analytic (everywhere) $\varphi\left(\frac{\pi}{2} + \pi k\right) = (-1)^{k+1}\left(\frac{\pi}{2} + \pi k\right)^3 \neq 0 \quad \forall k \in \mathbb{Z}$.

Thus, for any $k \in \mathbb{Z}$ $z_k = \frac{\pi}{2} + \pi k$ is a simple zero.

2) $\cos z^3 = 0 \Leftrightarrow z^3 = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$. Let us find all the roots of this equation:

$\left|\frac{\pi}{2} + \pi k\right| = \frac{\pi}{2} + \pi k$, $\arg\left(\frac{\pi}{2} + \pi k\right) = 0 + 2\pi l = 2\pi l$

$\Rightarrow z = \sqrt[3]{\frac{\pi}{2} + \pi k}\left(\cos\left(\frac{2\pi l}{3}\right) + i \sin\left(\frac{2\pi l}{3}\right)\right) \quad l = 0, 1, 2$

$l=0$: $\cos \frac{2\pi l}{3} + i \sin \frac{2\pi l}{3} = \cos 0 + i \sin 0 = 1$

$l=1$: $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$

$l=2$ $\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$

$\Rightarrow$ we have 3 solutions: $z_1 = \sqrt[3]{\frac{\pi}{2} + \pi k}$, $z_2 = \sqrt[3]{\frac{\pi}{2} + \pi k}\left(-\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$, and $z_3 = \sqrt[3]{\frac{\pi}{2} + \pi k}\left(-\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)$, $k \in \mathbb{Z}$. All these zeros are simple, namely of order 1.

b) $f(z) = \frac{z^5}{1 + z - e^z} \underset{\downarrow}{=} \frac{z^5}{1 + z - 1 - z - \frac{z^2}{2!} - \frac{z^3}{3!} - \dots} = \frac{z^5}{-\frac{z^2}{2!} - \frac{z^3}{3!} - \frac{z^4}{4!} - \frac{z^5}{5!} \dots}$

Use Maclaurin series for e^z

$$= \frac{z^3}{-\frac{1}{2!} - \frac{z}{3!} - \frac{z^2}{4!} - \dots} = z^3 \left(- \frac{1}{\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots} \right) = z^3 \varphi(z)$$

where $\varphi(z) = -\dfrac{1}{\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\dots}$ is analytic and $\varphi(0) = -2! = -2 \neq 0$

$\Rightarrow$ by Prop 16.3 $z=0$ is a zero of order 3.

2) $f(z) = 6\sin(z^3) + z^9 - 6z^3$. We use Maclaurin series for $\sin z$:

$\sin(z^3) = \left(z^3 - \frac{z^9}{3!} + \frac{z^{15}}{5!} - \frac{z^{21}}{7!} \dots\right)$

$\Rightarrow f(z) = 6z^3 - z^9 + \frac{6z^{15}}{5!} - \frac{6z^{21}}{7!} + \dots + z^9 - 6z^3 = \frac{6z^{15}}{5!} - \frac{6z^{21}}{7!} + \frac{6z^{27}}{9!} \dots$

$= z^{15}\left(\frac{6}{5!} - \frac{6z^6}{7!} + \frac{6z^{12}}{9!} - \dots\right) = z^{15}\varphi(z)$,

where $\varphi(z)$ is analytic and $\varphi(0) = \frac{6}{5!} \neq 0 \Rightarrow$ by Prop 16.3 $z=0$ is a zero of order 15.

3) $f(z) = (e^{z^2} - 1 - z^2)\sin^3 z$

First, we use Maclaurin series for e^{z^2} to obtain:

$e^{z^2} - 1 - z^2 = 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \dots - 1 - z^2 = \frac{z^4}{2!} + \frac{z^6}{3!} + \dots$

Now we use Maclaurin series for $\sin z$:

$\sin^3 z = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)^3 = z^3 - \frac{z^5}{2!} + \dots$

$\Rightarrow (e^{z^2} - 1 - z^2)\sin^3 z = \left(\frac{z^4}{2!} + \frac{z^6}{3!} + \dots\right)\left(z^3 - \frac{z^5}{2!} + \dots\right) =$

$= z^4\left(\frac{1}{2!} + \frac{z^2}{3!} + \dots\right) z^3\left(1 - \frac{z^2}{2!} + \dots\right) = z^7\left(\frac{1}{2!} + \frac{z^2}{3!} + \dots\right)\left(1 - \frac{z^2}{2!} + \dots\right)$

$= z^7\varphi(z)$

$\varphi(z)$ is analytic and $\varphi(0) = \frac{1}{2!} = \frac{1}{2} \neq 0 \Rightarrow z=0$ is a zero of order 7.

c) $f(z) + g(z)$: z_0 is a zero of order n of $f \Rightarrow$ by Prop 16.3

$f(z) = (z-z_0)^n \varphi(z)$, $\varphi(z)$ is analytic, $\varphi(z_0) \neq 0$

z_0 is a zero of order m of $g \Rightarrow$ by Prop 16.3

$g(z) = (z-z_0)^m \psi(z)$, $\psi(z)$ is analytic, $\psi(z_0) \neq 0$

$\Rightarrow f(z) + g(z) = (z-z_0)^n\varphi(z) + (z-z_0)^m\psi(z) \underset{\circledast}{=} (z-z_0)^{\min(n,m)} \cdot$

$\cdot \left(\varphi(z) + (z-z_0)^{m-n}\psi(z)\right)$

$\circledast$ Assume $n<m$; if $n>m$ get $\left(\psi(z) + (z-z_0)^{n-m}\varphi(z)\right)$

Since $\varphi(z_0) \neq 0$ (or for $n>m$ $\psi(z_0) \neq 0$) by Prop 16.3 z_0 is a zero of order $\min(n,m)$ of $f(z)+g(z)$.

$f(z)g(z) = (z-z_0)\varphi(z)(z-z_0)^m\psi(z) = (z-z_0)^{n+m}\varphi(z)\psi(z)$,

where $(\varphi\psi)(z_0) \neq 0$ and $\varphi \cdot \psi$ is an analytic function. Thus, by Prop 16.3 z_0 is a zero of order $n+m$ of $f \cdot g$

$\dfrac{f(z)}{g(z)} = \dfrac{(z-z_0)^n\varphi(z)}{(z-z_0)^m\psi(z)}$: if $n>m$ then $\frac{f}{g}$ has a zero of order $n-m$; if $n=m$, then $\frac{f}{g}$ does not have zero at z_0;

if $n<m$, then $n-m<0$, namely $\frac{f}{g}$ has a <u>pole</u> of order $m-n$ at z_0 and does not have zero at z_0.

d) Pf: $z=z_0$ is a zero of order n of $f(z)$ $\Rightarrow$ by Prop 16.3 $f(z)=(z-z_0)^n\varphi(z)$, where $\varphi(z)$ is analytic and $\varphi(z_0)\neq 0$. Thus, $(f(z))^2=((z-z_0)^n\varphi(z))^2=(z-z_0)^{2n}\varphi^2(z)$.
Since $\varphi^2(z_0)=\varphi(z_0)\varphi(z_0)\neq 0$ by Prop 16.3 z_0 is a zero of order $2n$ of $(f(z))^2$.

2) a) 1) $z\bar{z}+i(z-\bar{z})-2=0$. Let $z=x+iy$. Then: $z\bar{z}=x^2+y^2$, $z-\bar{z}=2iy$, and $z\bar{z}+i(z-\bar{z})-2=x^2+y^2-2y-2=0$.
Let us complete the square in y and get: $x^2+(y-1)^2=3$ - this is an equation of a circle centered at $(0,1)$ of radius 3. We rewrite: $\left(\frac{x}{\sqrt{3}}\right)^2+\left(\frac{y-1}{\sqrt{3}}\right)^2=1$: take $x=\sqrt{3}\cos t$, $y=\sqrt{3}\sin t+1$ for $0\le t<2\pi$. The parametric representation then is:

$$z(t)=\sqrt{3}\cos t+i(\sqrt{3}\sin t+1)$$

2) $\operatorname{Im}\left(\frac{1}{z}\right)=\frac{1}{2}$: $\frac{1}{z}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2} \Rightarrow \operatorname{Im}\frac{1}{z}=-\frac{y}{x^2+y^2}$
$\Rightarrow \operatorname{Im}\frac{1}{z}=\frac{1}{2}$: $-\frac{y}{x^2+y^2}=\frac{1}{2} \Leftrightarrow -2y=x^2+y^2 \Leftrightarrow x^2+y^2+2y=0$
$\Leftrightarrow x^2+(y+1)^2=1$.
Take $x=\cos t$, $y=\sin t-1$, $0\le t<2\pi$. Then, we obtain

$$z(t)=\cos t+i(\sin t-1),\quad 0\le t<2\pi$$

b) Let us look at each of the expressions for $z=x+iy$:
$\operatorname{Im} z=0 \Rightarrow y=0 \Rightarrow z=x$
$|\arg z-\frac{\pi}{2}|=\frac{\pi}{2} \Rightarrow \arg z-\frac{\pi}{2}=\frac{\pi}{2}$ or $\frac{\pi}{2}-\arg z=\frac{\pi}{2} \Rightarrow \arg z=2\pi k$ or $\arg z=\pi+2\pi k$. Let $z=re^{i\theta}$, $r>0$.
For $\arg z=\pi+2\pi k$: $z=re^{i(\pi+2\pi k)}=re^{-i\pi}=-r<0$ - the negative half of the real axis. For $\arg z=2\pi k$ we get $z=re^{i2\pi k}=r>0$ - the positive half of the real axis.
Combining: $|\arg z-\frac{\pi}{2}|=\frac{\pi}{2}$ describes the real axis.
$z-\bar{z}=x+iy-(x-iy)=2iy=0 \Leftrightarrow y=0 \Leftrightarrow z=x$
$|z-i|=|z+i| \Leftrightarrow |x+i(y-1)|=|x+i(y+1)| \Leftrightarrow$
$\sqrt{x^2+(y-1)^2}=\sqrt{x^2+(y+1)^2} \Leftrightarrow x^2+(y-1)^2=x^2+(y+1)^2 \Leftrightarrow$ for all x
$(y-1)^2=(y+1)^2 \Leftrightarrow$ for all x $y^2-2y+1=y^2+2y+1 \Leftrightarrow$ for all x
$4y=0 \Leftrightarrow$ for all x $y=0 \Rightarrow z=x$.

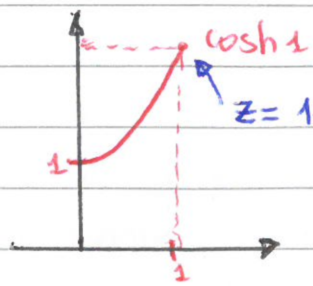
3) a) $f(z)=\frac{1/5}{z-2i}+\frac{1}{(z-2i)^2}+\frac{1}{(z-2i)^3}$

b) $f(z)=\frac{8(i-3)}{z-i+3}+6e^{\frac{1}{z-i}}$: $e^{\frac{1}{z-i}}=1+\frac{1}{z-i}+\frac{1}{2!}\frac{1}{(z-i)^2}+\dots$

$\Rightarrow z = i$ is an essential singularity since the principal part of the laurent series is infinite.

c) $f(z) = \frac{1}{(z-4i+1)^4} + \frac{7}{(z-4i+1)^3} + \frac{5}{z-4i+1} - \frac{3}{4}\cdot\frac{1}{z+i}$

4) a)



$z = 1 + i\cosh 1$ $\qquad \cosh z = \frac{e^z + e^{-z}}{2}$

b) $\gamma : [0,1] \to \mathbb{C}$ could have parametrization $t \to t + i\cosh t, 0 \le t \le 1$.

c) $L = \text{length of } \gamma = \int_a^b |\gamma'(t)|\,dt = \int_0^1 |1 + i\sinh t|\,dt =$

$= \int_0^1 \sqrt{1+\sinh^2 t}\,dt = \int_0^1 \cosh t\,dt = \sinh 1 - \underbrace{\sinh 0}_{=0} = \sinh 1$

5) $f(z) = 1 + i - 2\bar{z}$ from $z_0 = 0$ to $z_1 = 1+i$ along:

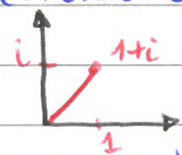
1) Straight line segment: Let us parametrize - for example, $\gamma(t) = (1+i)t$, $0 \le t \le 1$. For $t=0$: $\gamma(0) = 0$, $\gamma(1) = 1+i$. Thus,

$\int_C f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 (1+i - 2\overline{(1+i)}t)(1+i)\,dt =$

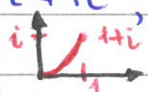
$= \int_0^1 (1+i - 2(1-i)t)(1+i)\,dt = \int_0^1 (1+i)^2 - t2(1-i)(1+i)\,dt =$

$= \int_0^1 (2i - 4t)\,dt = 2it - \frac{4t^2}{2}\Big|_0^1 = 2i - 2$

The path is:



2) $y = x^2$: Parametrization: $\gamma(t) = t + it^2$, $0 \le t \le 1$, $\gamma(0) = 0$, $\gamma(1) = 1+i$, $\gamma'(t) = 1 + 2it$, and



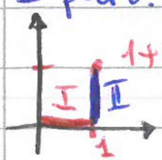
$\int_C f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 (1+i - 2\overline{(t+it^2)})(1+2it)\,dt$

$= \int_0^1 (1+i - 2t + 2it^2)(1+2it)\,dt = \int_0^1 (1+i - 4t + 2it - 2it^2 - 4t^3)\,dt$

$= ((1+i)t - 2t^2 + it^2 - \frac{2it^3}{3} - t^4)\Big|_0^1 = 1+i-2+i-\frac{2}{3}i - 1 = -2 + i\frac{4}{3}$

3) Parametrization: $z_2 = 1 \Rightarrow$ the first part of the path is a straight segment connecting $(0,0)$ and $(1,0)$ and the second part is a straight line segment connecting $z_2 = (1,0)$ and $z_1 = (1,1) = 1+i$

I part: $\gamma_1(t) = t$, $0 \le t \le 1$ $\qquad$ II part $\gamma_2(t) = 1 + it$, $0 \le t \le 1$



$\int_C f(z)\,dz = \int_I f(\gamma_1(t))\gamma_1'(t)\,dt + \int_{II} f(\gamma_2(t))\gamma_2'(t)\,dt$

$= \int_0^1 (1+i-2\bar{t})\cdot 1\,dt + \int_0^1 (1+i - 2\overline{(1+it)})\,i\,dt$

$= \int_0^1 (1+i-2t)\,dt + \int_0^1 (1+i-2+2it)\,i\,dt = (1+i)t - t^2\Big|_0^1$

$+ ((-i-1)t - t^2)\Big|_0^1 = 1+i-1-i-1-1 = -2$