121

Problem Set 7 - Solutions.

1) a) 1) $Ln(-i) = \ln|-i| + i(\arg(-i) + 2\pi k) = \ln 1 + i(\frac{3\pi}{2} + 2\pi k)$
$= i(\frac{3\pi}{2} + 2\pi k)$

2) $Ln(i^i)$: In Prop 15.1 we have proved that for any $n \in \mathbb{N}$ $Ln z^n = n Ln z$. Here instead of n we have i, thus we need to be more careful. Assume that $Ln i^i = i Ln i$, now let us compute $Ln i$ first:

$$Ln i = \ln|i| + i \arg(i) = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

We haven't added $2\pi k$ since we need to add it to the final answer, namely

$$Ln i^i = i Ln i = i(i\frac{\pi}{2} + 2\pi k) = -\frac{\pi}{2} + 2\pi k i$$

Note: Since $i^i = e^{Ln i^i} = e^{-\pi/2 + 2\pi k i} = e^{-\frac{\pi}{2}} e^{2\pi k i} = e^{-\pi/2}$

we conclude that $i^i = e^{-\pi/2} \in \mathbb{R}$! $\quad e^{2\pi k i} = 1 \quad \forall k$

3) $Ln(3-2i) = \ln|3-2i| + i(\arg(3-2i) + 2\pi k) = \ln\sqrt{9+4} + i\arg(3-2i) + 2\pi k i = \frac{1}{2}\ln(13) + i\arg(3-2i) + 2\pi k i$.

As we have seen in class $\arg(x+iy) = \operatorname{arctg}\frac{y}{x} \Rightarrow$
$\arg(3-2i) = \operatorname{arctg}(-\frac{2}{3}) = -\operatorname{arctg}\frac{2}{3}$ ($\operatorname{arctg} z$ is an odd function)
$\Rightarrow Ln(3-2i) = \frac{1}{2}\ln 13 + i(2\pi k - \operatorname{arctg}\frac{2}{3})$

b) 1) Since $z^4 - 2 = (z^2 - \sqrt{2})(z^2 + \sqrt{2}) = (z - \sqrt[4]{2})(z + \sqrt[4]{2})(z - i\sqrt[4]{2}) \cdot (z + i\sqrt[4]{2})$,

by Prop 15.1 1) we have

$$Ln(z^4 - 2) = Ln(z - \sqrt[4]{2}) + Ln(z + \sqrt[4]{2}) + Ln(z - i\sqrt[4]{2}) + Ln(z + i\sqrt[4]{2})$$

$\Rightarrow$ the branch points are $-\sqrt[4]{2}, +\sqrt[4]{2}, -i\sqrt[4]{2}, +i\sqrt[4]{2}$.

2) By Prop 15.1 2) we get:

$$(z-8i) Ln\left(\frac{3z-i}{z+4}\right) - 4i = (z-8i)\left(Ln(3z-i) - Ln(z+4)\right) - 4i$$

$\Rightarrow$ the branch points are $\frac{i}{3}$ and -4.

c) By Prop 15.2 every branch of the function $Ln(z-1)$ is analytic in the cut-plane where the cut is made along any ray that starts at $z=1$. Therefore, in particular in the cut-plane with the cut along the negative half of the real axis up to (including) $x=1$. We also need to deal with the zeros of the denominator: $z^2 + i = 0 \Leftrightarrow z^2 = -i \Leftrightarrow z = \pm\sqrt{-i}$.

Let us compute $\sqrt{-i}$: $|-i|=1$, $\arg(-i) = \frac{3\pi}{2} + 2\pi k$. Thus:

$$\sqrt{-i} = 1\left(\cos\left(\frac{\frac{3\pi}{2}+2\pi k}{2}\right) + i\sin\left(\frac{\frac{3\pi}{2}+2\pi k}{2}\right)\right), \quad k=0,1$$

$k=0$: $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt2} + i\frac{1}{\sqrt2} = -\frac{1}{\sqrt2}(1-i)$

$k=1$ $\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt2} - i\frac{1}{\sqrt2} = \frac{1}{\sqrt2}(1-i)$

Therefore, if we puncture the points $z = \pm\frac{1}{\sqrt2}(1-i)$, then the given function is analytic in the cut-plane with these 2 punctured points.

2) a) $\arccos z = -i\,\mathrm{Ln}(z \pm i\sqrt{1-z^2})$:

$w = \arccos z \Rightarrow z = \cos w = \frac12(e^{iw}+e^{-iw})$. Denote $e^{iw}=t$, then $z = \frac12(t+\frac1t) \Leftrightarrow t^2 - 2tz + 1 = 0 \Rightarrow t_{1,2} = z \pm \sqrt{z^2-1}$ or $t_{1,2} = z \pm i\sqrt{1-z^2}$. Namely, $e^{iw} = z \pm i\sqrt{1-z^2}$, thus $iw = \mathrm{Ln}(z \pm i\sqrt{1-z^2}) \Rightarrow w = \frac1i \mathrm{Ln}(z\pm i\sqrt{1-z^2}) = -i\,\mathrm{Ln}(z \pm i\sqrt{1-z^2})$

$\mathrm{arctg}\, z = \frac{1}{2i}\mathrm{Ln}\left(\frac{1+iz}{1-iz}\right)$:

First: $\mathrm{tg}\, z = \frac{\sin z}{\cos z}$, thus $\mathrm{tg}\, z = \frac{\frac{1}{2i}(e^{iz}-e^{-iz})}{\frac12(e^{iz}+e^{-iz})} = -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

for all $z \in \mathbb{C}$. So, if $w = \mathrm{arctg}\, z \Rightarrow$

$$z = \mathrm{tg}\, w = -i\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} = -i\frac{e^{2iw}-1}{e^{2iw}+1}$$

Denote $t = e^{iw}$, then: $z = \frac{i-it^2}{t^2+1} \Rightarrow iz = \frac{t^2-1}{t^2+1} \Rightarrow t^2 = \frac{1+iz}{1-iz}$

$\Rightarrow e^{2iw} = \frac{1+iz}{1-iz} \Rightarrow 2iw = \mathrm{Ln}\left(\frac{1+iz}{1-iz}\right) \Rightarrow w = \frac{1}{2i}\mathrm{Ln}\left(\frac{1+iz}{1-iz}\right)$

b) 1) $\arcsin(\frac{\pi}{3}i) = -i\,\mathrm{Ln}\left(i\cdot\frac{\pi}{3}i \pm \sqrt{1-(\frac{\pi}{3}i)^2}\right) = -i\,\mathrm{Ln}\left(-\frac{\pi}{3} \pm \sqrt{1+\frac{\pi^2}{9}}\right)$

Let us compute each of the logarithms:

$\mathrm{Ln}\left(-\frac{\pi}{3} + \sqrt{1+\frac{\pi^2}{9}}\right) = \ln\left|-\frac{\pi}{3} + \sqrt{1+\frac{\pi^2}{9}}\right| + i\left(\arg\left(-\frac{\pi}{3}+\sqrt{1+\frac{\pi^2}{9}}\right) + 2\pi k\right)$

$= \ln\left(-\frac{\pi}{3} + \frac13\sqrt{9+\pi^2}\right) + 2\pi k i$ $\left\{-\frac{\pi}{3} + \frac13\sqrt{9+\pi^2} > 0 \Rightarrow \arg\left(-\frac{\pi}{3}+\frac13\sqrt{9+\pi^2}\right) = 0\right\}$

$\Rightarrow -i\,\mathrm{Ln}\left(-\frac{\pi}{3} + \sqrt{1+\frac{\pi^2}{9}}\right) = 2\pi k - i\ln\left(-\frac{\pi}{3} + \frac13\sqrt{9+\pi^2}\right)$

$\mathrm{Ln}\left(-\frac{\pi}{3} - \sqrt{1+\frac{\pi^2}{9}}\right) = \ln\left|-\frac{\pi}{3} - \sqrt{1+\frac{\pi^2}{9}}\right| + i\left(\arg\left(-\frac{\pi}{3} - \sqrt{1+\frac{\pi^2}{9}}\right) + 2\pi k\right)$

$= \ln\left(\frac{\pi}{3} + \sqrt{1+\frac{\pi^2}{9}}\right) + i(\pi + 2\pi k)$ $\left\{-\frac{\pi}{3} - \sqrt{1+\frac{\pi^2}{9}} < 0 \Rightarrow \arg\left(-\frac{\pi}{3} - \sqrt{1+\frac{\pi^2}{9}}\right) = \pi\right\}$

$\Rightarrow -i\,\mathrm{Ln}\left(-\frac{\pi}{3} - \sqrt{1+\frac{\pi^2}{9}}\right) = \pi(2k+1) - i\ln\left(\frac{\pi}{3} + \sqrt{1+\frac{\pi^2}{9}}\right)$

2) $\mathrm{arctg}(1+i) = \frac{1}{2i}\mathrm{Ln}\left(\frac{1+i(1+i)}{1-i(1+i)}\right) = \frac{1}{2i}\mathrm{Ln}\left(\frac{i}{2-i}\right) = \frac{1}{2i}\mathrm{Ln}\left(\frac{i(2+i)}{5}\right)$

$= \frac{1}{2i}\mathrm{Ln}\left(\frac{2i-1}{5}\right) = \frac{1}{2i}\left(\mathrm{Ln}(2i-1) - \mathrm{Ln}\,5\right) = \frac{1}{2i}\left(\ln|2i-1| + i(\arg(2i-1) +\right.$

$+2\pi k) - \ln 5) = \frac{1}{2i}\left(\ln\sqrt{4+1} + i\,\mathrm{arctg}\frac{2}{-1} + 2\pi k i - \ln 5\right)$

$= \frac{1}{2i}\left(\frac{1}{2}\ln 5 - \ln 5 + i\,\mathrm{arctg}(-2) + 2\pi k i\right) =$

$= \frac{1}{2i}\left(-\frac{1}{2}\ln 5 - i\,\mathrm{arctg}\,2 + 2\pi k i\right) = \frac{i}{4}\ln 5 - \frac{1}{2}\mathrm{arctg}\,2 + \pi k$

c) 1) $\sin z = 5 \Rightarrow z = \arcsin 5$. Use the formula

$$\arcsin z = -i\,\mathrm{Ln}\left(iz \pm \sqrt{1-z^2}\right)$$

$\arcsin 5 = -i\,\mathrm{Ln}\left(5i \pm \sqrt{1-25}\right) = -i\,\mathrm{Ln}\left(5i \pm i\sqrt{24}\right) = -i\,\mathrm{Ln}\left((5\pm\sqrt{24})i\right)$

$= -i\left\{\ln(5\pm\sqrt{24}) + i\left(\frac{\pi}{2} + 2\pi k\right)\right\} = \frac{\pi}{2} + 2\pi k - i\ln(5\pm\sqrt{24})$

$5\pm\sqrt{24} > 0 \Rightarrow \arg(i(5\pm\sqrt{24})) = \frac{\pi}{2}$, $|i(5\pm\sqrt{24})| = |i||5\pm\sqrt{24}|$

2) $3\cos z - 7 = 0 \Rightarrow \cos z = \frac{7}{3} \Rightarrow z = \arccos\frac{7}{3}$

$\arccos z = -i\,\mathrm{Ln}\left(z \pm i\sqrt{1-z^2}\right)$ by 2a). Therefore:

$\arccos\left(\frac{7}{3}\right) = -i\,\mathrm{Ln}\left(\frac{7}{3} \pm i\sqrt{1-\frac{49}{9}}\right) = -i\,\mathrm{Ln}\left(\frac{7}{3} \pm i\sqrt{-\frac{40}{9}}\right)$

$= -i\,\mathrm{Ln}\left(\frac{7}{3} \pm i\cdot\left(\pm i\frac{\sqrt{40}}{3}\right)\right)$

We have 4 cases:

$+i\left(+i\frac{\sqrt{40}}{3}\right) = -\frac{\sqrt{40}}{3} \Rightarrow$ get $-i\,\mathrm{Ln}\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right)$ $\left\{\frac{7}{3} - \frac{\sqrt{40}}{3} > 0!\right\}$

$+i\left(-i\frac{\sqrt{40}}{3}\right) = \frac{\sqrt{40}}{3} \Rightarrow$ get $-i\,\mathrm{Ln}\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right)$

$-i\left(+i\frac{\sqrt{40}}{3}\right) = \frac{\sqrt{40}}{3} \Rightarrow -i\,\mathrm{Ln}\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right)$

$-i\left(-i\frac{\sqrt{40}}{3}\right) = -\frac{\sqrt{40}}{3} \Rightarrow -i\,\mathrm{Ln}\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right)$

$\mathrm{Ln}\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right) = \ln\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right) + i(0 + 2\pi k) = \ln\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right) + 2\pi k i$

$\mathrm{Ln}\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right) = \ln\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right) + i(0 + 2\pi k) = \ln\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right) + 2\pi k i$

The answer is:

$-i\left(\ln\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right) + 2\pi k i\right) = 2\pi k - i\ln\left(\frac{7}{3} - \frac{\sqrt{40}}{3}\right)$

$-i\left(\ln\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right) + 2\pi k i\right) = 2\pi k - i\ln\left(\frac{7}{3} + \frac{\sqrt{40}}{3}\right)$

d+e) $\mathrm{arcsinh}\, z$ is $w \in \mathbb{C}$ s.t. $\sinh w = z$. By definition of $\sinh w$:

$z = \frac{1}{2}(e^w - e^{-w})$. Denote $t = e^w \Rightarrow z = \frac{t^2-1}{2t} \Rightarrow t^2 - 2tz - 1 = 0$

$\Rightarrow t_{1,2} = z \pm \sqrt{z^2+1} \Rightarrow e^w = z \pm \sqrt{z^2+1} \Rightarrow w = \mathrm{Ln}\left(z \pm \sqrt{z^2+1}\right)$

$\mathrm{arccosh}\, z$ is $w \in \mathbb{C}$ s.t. $\cosh w = z$. By definition of $\cosh w$:

$z = \frac{1}{2}(e^w + e^{-w})$. Denote $t = e^w \Rightarrow z = \frac{t^2+1}{2t} \Rightarrow t^2 - 2tz + 1 = 0$

$\Rightarrow t_{1,2} = z \pm \sqrt{z^2-1} \Rightarrow e^w = z \pm \sqrt{z^2-1} \Rightarrow w = \mathrm{Ln}\left(z \pm \sqrt{z^2-1}\right)$

3) a) The singularities of f are the zeros of the denominator:

$z^3 + z^2 - z - 1 = 0$: Guess one root $z = 1$, then divide:

$$\begin{array}{l|l} z^3+z^2-z-1 & z-1 \\ \hline z^3-z^2 & z^2+2z+1 \\ 2z^2-z & \\ 2z^2-2z & \\ z-1 & \\ z-1 & \\ 0 & \end{array}$$

$\Rightarrow z^3 + z^2 - z - 1 = (z-1)(z^2+2z+1) = (z-1)(z+1)^2$

$\Rightarrow$ the singularities are $z = 1$, $z = -1$

Start with $z=1$. Rewrite $f(z) = \frac{\varphi(z)}{(z+1)^2}$, where $\varphi(z) = \frac{\sin(z-1)}{(z-1)}$

$\frac{1}{(z+1)^2}$ is analytic at $z=1$ and we have $\lim_{z\to 1} (z-1)\frac{\sin(z-1)}{(z-1)} = 0$

$\Rightarrow$ by Prop 16.1 $z=1$ is a removable singularity.

For $z=-1$: $f(z) = \frac{\sin(z-1)}{(z-1)} \cdot \frac{1}{(z+1)^2}$; $\varphi(z)$ is analytic at $z=-1$ and its neighborhood and $\varphi(-1) = \frac{\sin(-2)}{-2} \neq 0$. Thus, $f(z)$ has a pole of order 2 at $z=-1$.

Let us compute the residue at $z=-1$ [no residue at $z=1$, since it is a removable singularity]

By Prop 16.2:

$$\mathrm{Res}(f;-1) = \frac{\varphi^{(m-1)}(-1)}{(m-1)!} \overset{m=2}{=} \frac{\varphi'(-1)}{1!} = \frac{-2\cos(-2)-\sin(-2)}{4} =$$

$$= \frac{1}{4}(\sin 2 - 2\cos 2)$$

b+c) 1) $$\frac{z^2-1}{z^6+2z^5+z^4} = \frac{z^2-1}{z^4(z^2+2z+1)} = \frac{(z-1)(z+1)}{z^4(z+1)^2} \underset{z\neq -1}{=} \frac{z-1}{z^4(z+1)}$$

Thus, there are 2 isolated singularities $z=0$ and $z=-1$.

$z=0$: Rewrite $f(z) = \frac{\varphi(z)}{z^4}$, where $\varphi(z) = \frac{z-1}{z+1}$. φ is analytic in a neighborhood of $z=0$ (and at $z=0$) and $\varphi(0) = -1 \neq 0 \Rightarrow$ by Prop 16.2 $z=0$ is a pole of order $m=4$. By Prop 16.2:

$$\mathrm{Res}(f;0) = \frac{\varphi^{(3)}(0)}{3!} = \frac{12}{6} = 2$$

$z=-1$: Rewrite $f(z) = \frac{\tilde{\varphi}(z)}{z+1}$, where $\tilde{\varphi}(z) = \frac{z-1}{z^4}$ – it is analytic in a neighborhood of $z=-1$ (and at $z=-1$) $\tilde{\varphi}(-1) = -2 \neq 0 \Rightarrow$ by Prop 16.2 $z=-1$ is a simple pole (of order 1). By Prop 16.2:

$$\mathrm{Res}(f;-1) = \frac{\tilde{\varphi}(-1)}{0!} = -2$$

2) We use the Taylor series expansion for $\sin z$ about $z=0$ ($z=0$ is the only candidated for a singularity, since the denominator is zero at $z=0$)

$$\frac{z-\sin z}{z^7} = \frac{1}{z^7}\left(z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \frac{z^9}{9!} + \dots\right) = \frac{1}{3!}\frac{1}{z^4} - \frac{1}{5!}\frac{1}{z^2} + \frac{1}{7!} -$$

$$- \frac{z^2}{9!} + \dots) = \frac{1}{z^4}\left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \frac{z^6}{9!} + \dots\right) = \frac{1}{z^4}\varphi(z)$$

$\varphi(z)$ is analytic in a neighborhood of $z=0$ (and at $z=0$), and $\varphi(0) = \frac{1}{3!} \neq 0 \Rightarrow$ there is an isolated singularity at $z=0$ and by Prop 16.2 it is a pole of order 4. From expansion $\mathrm{Res}(f;0) = 0$

3) The singularities are the zeros of the denominator and the singularities (if there are any) of the numerator.

The denominator $e^z - 1 = 0 \Leftrightarrow e^z = 1 \overset{*}{\Leftrightarrow} z = 2\pi n i,\ n \in \mathbb{Z}$

$*$ If $z = 2\pi n i$, then by Euler's formula

$$e^{2\pi n i} = \cos(2\pi n) + i \sin(2\pi n) = 1$$

Assume now that $e^z = 1$. Then, by Euler's formula:

$e^x(\cos y + i \sin y) = 1 \quad (z = x+iy) \Rightarrow \begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0 \end{cases} \Leftrightarrow x = 0,\ y = 2\pi n$

Therefore, since

$**$ $e^{\frac{1}{2\pi n i - 1}} = e^{\frac{-2\pi n i - 1}{(1+4\pi^2 n^2)^{1/2}}} = e^{-\frac{1}{(1+4\pi^2n^2)^{1/2}}} \, e^{-i \frac{2\pi n}{(1+4\pi^2 n^2)^{1/2}}} =$

$= e^{-1/(1+4\pi^2 n^2)^{1/2}} \left(\cos \frac{2\pi n}{(1+4\pi^2 n^2)^{1/2}} - i \sin \frac{2\pi n}{(1+4\pi^2n^2)^{1/2}} \right) \neq 0$

we conclude that $f(z)$ has infinitely many simple poles at $z = 2\pi n i$, $n \in \mathbb{Z}$, and by Prop 16.2

$\mathrm{Res}(f, 2\pi n i) = **$

There is an additional singularity at $z = 1$ (in the numerator). It is an isolated singularity. To determine its nature we look at the Laurent series expansion about $z=1$:

$$e^{1/(z-1)} = e^{(z-1)^{-1}} = \sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^{-n} = 1 + \frac{1}{z-1} + \frac{1}{2!}\frac{1}{(z-2)^2} + \dots$$

Since the principal part consists of infinitely many non-zero terms by definition it is an essential singularity. From the expansion: $\mathrm{Res}(f; 1) = b_1 = 1$.

4) $\cosh^2 \frac{1}{z-\pi} = \left(\frac{1}{2}\left(e^{\frac{1}{z-\pi}} + e^{-\frac{1}{z-\pi}} \right) \right)^2 = \frac{1}{4}\left(e^{\frac{2}{z-\pi}} + e^{-\frac{2}{z-\pi}} + 2 \right)$

As in 3) we conclude that $z = \pi$ is an essential singularity. To determine the residere we look at the Laurent series expansion:

$$e^{\frac{2}{z-\pi}} = \left(e^{1/z-\pi} \right)^2 = \left(\sum_{n=0}^{\infty} \frac{1}{n!}(z-\pi)^{-n} \right)^2 = \left(1 + \frac{1}{z-\pi} + \frac{1}{2!}\frac{1}{(z-\pi)^2} + \dots\right) \cdot$$

$$\cdot \left(1 + \frac{1}{z-\pi} + \frac{1}{2!}\frac{1}{(z-\pi)^2} + \dots \right) = 1 + \frac{2}{z-\pi} + \dots$$

In the same way:

$$e^{-\frac{2}{z-\pi}} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z-\pi)^{-n} \right)^2 = \left(1 - \frac{1}{z-\pi} + \frac{1}{2!}\frac{1}{(z-\pi)^2} - \dots \right)^2$$

$$= 1 - \frac{2}{z-\pi} + \dots$$

$\Rightarrow \frac{1}{4}\left(e^{2/z-\pi} + e^{-2/z-\pi} + 2 \right) = \frac{1}{4}\left(2 + 1 + \frac{2}{z-\pi} + \dots + 1 - \frac{2}{z-\pi} + \dots \right)$

$= 1 + \frac{1}{(z-\pi)^2} - \dots$

$\Rightarrow \mathrm{Res}(f; \pi) = 0$.

5) $\frac{\sinh(z^4) - z}{z^6}$: Recall: about $z=0$ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

$z = 0$ is the only candidate for a singularity.

$$\frac{\sinh(z^4)-z}{z^6} = \frac{1}{z^6}\left(z^4+\frac{z^{12}}{3!}+\frac{z^{20}}{5!}+\dots - z\right) = -\frac{1}{z^5}+\frac{1}{z^2}+\frac{z^6}{3!}+\dots$$

$$= \frac{1}{z^5}\left(-1+z^3+\frac{z^{11}}{3!}+\dots\right) = \frac{1}{z^5}\varphi(z)$$

$\varphi(z)$ is analytic in a neighborhood of $z=0$ (and at $z=0$), and $\varphi(0)=-1\neq 0 \Rightarrow$ by Prop 16.2 $z=0$ is a pole of order 5.
We see from the expansion that $\operatorname{Res}(f,0)=0$.

6) $z^5\cosh\frac{1}{z^2}$: the only candidate for singularity is $z=0$ and about $z=0$ $\cosh z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} \Rightarrow \cosh\frac{1}{z^2} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{1}{z^2}\right)^{2n}$

$$\Rightarrow z^5\cosh\frac{1}{z^2} = z^5\left(1+\frac{1}{2!}\frac{1}{z^4}+\frac{1}{4!}\frac{1}{z^8}+\frac{1}{6!}\frac{1}{z^{12}}+\dots\right) =$$

$$= z^5+\frac{1}{2!}z+\frac{1}{4!}\frac{1}{z^3}+\frac{1}{6!}\frac{1}{z^7}+\dots$$

$\Rightarrow$ there is an isolated singularity at $z=0$ and since the principal part has infinitely many non-zero terms, it is an essential singularity. From the expansion we see that $\operatorname{Res}(f,0)=0$.

7) $\frac{e^{-iz}}{z^2-\frac{\pi^2}{4}} = \frac{e^{-iz}}{(z-\frac{\pi}{2})(z+\frac{\pi}{2})} \Rightarrow$ 2 isolated singularities: $z=\pm\frac{\pi}{2}$

$z=\frac{\pi}{2}$: Rewrite $f(z)=\frac{1}{z-\frac{\pi}{2}}\varphi(z)$, $\varphi(z)=\frac{e^{-iz}}{z+\pi/2}$ is analytic in a neighborhood of $z=\frac{\pi}{2}$ (and at $z=\frac{\pi}{2}$), and

$$\varphi\left(\frac{\pi}{2}\right) = \frac{e^{-i\pi/2}}{\frac{\pi}{2}+\frac{\pi}{2}} = \frac{1}{\pi}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right) = -\frac{i}{\pi}\neq 0$$

$\Rightarrow$ by Prop 16.2 $z=\frac{\pi}{2}$ is a simple pole, and $\operatorname{Res}(f;\frac{\pi}{2})=-\frac{i}{\pi}$.
In the same way $z=-\frac{\pi}{2}$ is a simple pole, $\operatorname{Res}(f;-\frac{\pi}{2})=\frac{i}{\pi}$.

8) $\frac{1-\cosh z}{z^5}$: the only candidate for singularity is $z=0$

$$\Rightarrow \frac{1-\cosh z}{z^5} = \frac{1}{z^5}\left(1-1-\frac{z^2}{2!}-\frac{z^4}{4!}-\frac{z^6}{6!}-\dots\right) = -\frac{1}{2!}\frac{1}{z^3}-\frac{1}{4!}\frac{1}{z}-$$

$$-\frac{z}{6!}-\frac{z^3}{8!}-\dots = -\frac{1}{z^3}\left(\frac{1}{2!}+\frac{z^2}{4!}+\frac{z^4}{6!}+\frac{z^6}{8!}+\dots\right) = -\frac{1}{z^3}\varphi(z)$$

$\varphi(z)$ is analytic in a neighborhood of $z=0$ (and at $z=0$) and $\varphi(0)=+\frac{1}{2!}\neq 0 \Rightarrow$ by Prop 16.2 $z=0$ is a pole of order 3.
From the expansion we obtain $\operatorname{Res}(f;0)=\frac{1}{4!}=\frac{1}{24}$.

4) $f(z)=\frac{1}{(z-4)(z+8i)} = \left(-\frac{1}{20}+i\frac{1}{10}\right)\left(\frac{1}{z+8i}-\frac{1}{z-4}\right)$

a) $f(z)$ can be expanded into Taylor series of the form $\sum_{n=0}^{\infty}a_n(z-1)^n$:

$$\frac{1}{z+8i} = \frac{1}{1+8i+(z-1)} = \frac{1}{1+8i}\cdot\frac{1}{1+\left(\frac{z-1}{1+8i}\right)} = \frac{1}{1+8i}\cdot\frac{1}{1-\left(-\frac{z-1}{1+8i}\right)} =$$

$= \frac{1}{1+8i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{1+8i}\right)^n$ – geometric series, valid on $\left|\frac{z-1}{1+8i}\right|<1$,

(127)

namely, $|z-1| < |1+8i| = \sqrt{65}$.

$$\frac{1}{z-4} = \frac{1}{(z-1)-3} = -\frac{1}{3-(z-1)} = -\frac{1}{3}\cdot\frac{1}{1-\left(\frac{z-1}{3}\right)} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z-1}{3}\right)^n$$

- geometric series, valid on $|z-1|<3$

Therefore, the radius of convergence is $R=3$.

b) Laurent series of $f(z)$ about $z_0=-8i$ is given by

$$\sum_{n=0}^{\infty} a_n(z+8i)^n + \sum_{n=1}^{\infty} b_n(z+8i)^{-n}$$

$$f(z) = \left(-\frac{1}{20}+i\frac{1}{10}\right)\left(\frac{1}{z+8i} - \frac{1}{z-4}\right) = \left(-\frac{1}{20}+i\frac{1}{10}\right)\left(\frac{1}{z+8i} + \frac{1}{4+8i-(z+8i)}\right)$$

$$= \left(-\frac{1}{20}+i\frac{1}{10}\right)\left(\frac{1}{z+8i} + \frac{1}{4+8i}\cdot\frac{1}{1-\left(\frac{z+8i}{4+8i}\right)}\right) =$$

$$= \left(-\frac{1}{20}+i\frac{1}{10}\right)\left(\frac{1}{z+8i} + \frac{1}{4+8i}\underbrace{\sum_{n=0}^{\infty}\left(\frac{z+8i}{4+8i}\right)^n}_{\text{valid on } 0<|z+8i|<|4+8i|=\sqrt{80}}\right)$$

$$= \underbrace{\frac{-\frac{1}{20}+i\frac{1}{10}}{z+8i}}_{\text{the principal part}} + \underbrace{\left(\sum_{n=0}^{\infty}\frac{(z+8i)^n}{(4+8i)^{n+1}}\right)}_{\text{Radius of convergence } = \sqrt{80}}\left(-\frac{1}{20}+i\frac{1}{10}\right)$$

$\Rightarrow z_0=-8i$ is a simple pole and $\mathrm{Res}(f;-8i) = -\frac{1}{20}+i\frac{1}{10}$