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Week 8. Lecture 19.

Function Ln z.

Def 15.1: The natural logarithm of a complex number z is a complex number w such that $e^w = z$. Notation: $w = \mathrm{Ln}\, z$

The logarithm of $r \in \mathbb{R}$ denote as usual $\ln r$.

Let us develop a formula for the computation of the complex logarithm. Write: $z = r(\cos\varphi + i\sin\varphi)$, $w = u + iv$

By Def 15.1: $e^{u+iv} = r(\cos\varphi + i\sin\varphi) \Rightarrow e^u(\cos v + i\sin v) = r(\cos\varphi + i\sin\varphi)$

By comparing the complex numbers, we conclude:

$$e^u = r \Rightarrow u = \ln r, \quad \begin{cases} \cos v = \cos\varphi \\ \sin v = \sin\varphi \end{cases} \Rightarrow v = \varphi + 2\pi k, \ k \in \mathbb{Z}$$

$\Rightarrow \mathrm{Ln}\, z = w = u + iv = \ln r + i(\varphi + 2\pi k)$ for $k \in \mathbb{Z}$.

Recall: $r = |z|$, $\varphi + 2\pi k = \arg z$, thus

$$(*) \quad \mathrm{Ln}\, z = \ln|z| + i\arg z$$

$\Rightarrow \mathrm{Ln}\, z$ is defined for any $z \neq 0$ and it is multi-valued function (for different values of k the function attains different values)

Ex 17.1: Compute: 1) $\mathrm{Ln}\, i$ 2) $\mathrm{Ln}(-4)$

Sol: 1) $\mathrm{Ln}\, i = \ln|i| + i(\arg i + 2\pi k) = \ln 1 + i\left(\frac{\pi}{2} + 2\pi k\right) = i\left(\frac{\pi}{2} + 2\pi k\right)$

2) $\mathrm{Ln}(-4) = \ln|-4| + i(\arg(-4) + 2\pi k) = \ln 4 + i(\pi + 2\pi k)$

Ex 17.2: Find an analytic function $f(z)$ such that

$$\mathrm{Re}\, f(z) = \mathrm{arctg}\frac{y}{x} \quad \text{and} \quad f(-i) = \frac{3\pi}{2} + 2i$$

Sol: In Ex 11.3 we have seen that

$$f(z) = \mathrm{arctg}\frac{y}{x} - \frac{i}{2}\ln(x^2+y^2) + ic$$

$\Rightarrow$ the only question is to determine the value of the constant c.

Let $z = x+iy$. Recall: $\arg z - \theta$ s.t. $\cos\theta = \frac{x}{|z|}$, $\sin\theta = \frac{y}{|z|}$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \mathrm{tg}\,\theta = \frac{y}{x} \Rightarrow \theta = \mathrm{arctg}\frac{y}{x} \Rightarrow \arg z = \mathrm{arctg}\frac{y}{x}$ (up to 2π),

$|z|^2 = x^2 + y^2$

$\Rightarrow f(z) = \arg z - \frac{i}{2}\ln|z|^2 + ic = -i(\ln|z| + i\arg z) + ic$, or

$f(z) = -i\,\mathrm{Ln}\, z + c_1$, where $c_1 \in \mathbb{C}$ is a constant, let us compute it:

$$\frac{3\pi}{2} + 2i = f(-i) = -i\,\mathrm{Ln}(-i) + c_1 = -i\left(0 + i\frac{3\pi}{2}\right) + c_1 = \frac{3\pi}{2} + c_1$$

$\Rightarrow c_1 = 2i$ and $f(z) = i\,\mathrm{Ln}\, z + 2i$.

Prop 15.1: For any $z_1, z_2 \in \mathbb{C}$, $n \in \mathbb{N}$:

1) $\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2$

2) $\mathrm{Ln}(z_1/z_2) = \mathrm{Ln}\, z_1 - \mathrm{Ln}\, z_2$

3) $\mathrm{Ln}\, z^n = n\,\mathrm{Ln}\, z$

4) $\mathrm{Ln}\sqrt[n]{z} = \frac{1}{n}\mathrm{Ln}\, z$

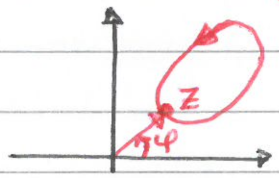
Pf: By formula ⊛, Prop 1.2, and Prop 1.3, we get:
$Ln(z_1 z_2) = \ln|z_1 z_2| + i\arg(z_1 z_2) = \ln(|z_1||z_2|) + i(\arg z_1 + \arg z_2)$
$= (\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2) = Ln\, z_1 + Ln\, z_2$
The rest in the same way (FINISH!) ☒

Rmk: If $z_1 = r_1 e^{i\varphi_1}$ $z_2 = r_2 e^{i\varphi_2}$, then 1) can be written as
$$Ln(z_1 z_2) = \ln r_1 + \ln r_2 + i\varphi, \text{ where } \varphi = \varphi_1 + \varphi_2 \pmod{2\pi}$$

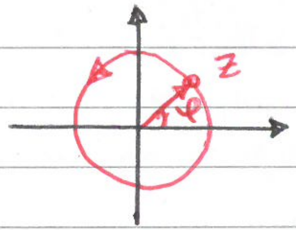
Q-n: CHECK – 2), 3), 4) !

Now let us study $Ln\, z$ in the same spirit as we studied the complex root

$w = Ln\, z$: For any z correspond infinite number of values w.
We choose a single-valued branch by fixing k_0: $w_{k_0} = \ln r + i(\varphi + 2\pi k_0)$



Let z move on some closed curve that does not encircle the origin, $z=0$. After completing a full circle, z returns to the same point with the same values of r and φ $\Rightarrow$ the point w_{k_0} does not change.



If z moves (anticlockwise) on some closed curve that encircles the origin, then when z returns to the starting point the modulus r is the same, but the argument of z will increase by 2π and $Ln\, z$ will attain a new value:

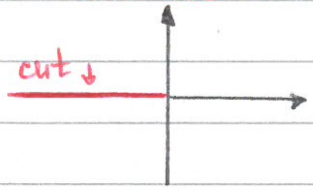
$w_{new} = \ln r + i((\varphi + 2\pi k_0) + 2\pi) = \ln r + i(\varphi + 2\pi(k_0+1)) = w_{k_0+1}$

So, after completing a full circle around the origin, the branch w_{k_0} "jumps" to a new branch w_{k_0+1}. If two full circles are completed: $Ln\, z$ will attain another (new) value: w_{k_0+2}, since the modulus is the same, but the argument will increase by 4π. In the same way, by completing finite number of full circles (anticlockwise) around the origin it is possible to pass from w_{k_0} branch of $Ln\, z$ to any given in advance branch. For example, after 7 full circles around the origin w_{k_0} will pass to w_{k_0+7} (if in anticlockwise direction) or to w_{k_0-7} (if clockwise direction)

The point $z=0$ is the branch point of the function $Ln\, z$.

Conclusion: In any domain that does not contain closed curves

that encircle $z=0$ one can choose infinitely many single-valued continuous branches of multi-valued function $w = \operatorname{Ln} z$.
If we cut the complex plane along any ray that starts at



$z=0$ (including $z=0$) - for example, the negative half of the real axis including $z=0$, we will get a cut-plane in which one can choose single-valued continuous branches. In this case the branch w_{k_0} maps the cut-plane z to a strip $\{w : k_0\pi < \operatorname{Im} w < (k_0+2)\pi\}$ in the w-plane.

CHECK: $z=z_0$ is the branch point of $\operatorname{Ln}(z-z_0)$ and in the cut-plane with the cut along some ray starting at $z=z_0$ (including z_0) one can choose infinitely many single-valued continuous branches.

<u>Prop 15.2</u>: Every branch of the function $\operatorname{Ln} z$ is an analytic function in the cut-plane (cutted along some ray starting at the origin) and $(\operatorname{Ln} z)' = \frac{1}{z}$.

<u>Pf</u>: We prove that $\operatorname{Ln} z$ is differentiable. Let Δz be such that z and $z+\Delta z$ are in the same domain. We compute: $\operatorname{Ln}(z+\Delta z) - \operatorname{Ln} z$. Since $e^w = z \Rightarrow e^{w+\Delta w} = z+\Delta z$, and

$$\frac{\Delta w}{\Delta z} = \frac{\operatorname{Ln}(z+\Delta z)-\operatorname{Ln} z}{\Delta z} \underset{(*)}{=} \frac{\Delta w}{\underbrace{e^{w+\Delta w}-e^w}_{=\Delta z}} = \frac{1}{\frac{e^{w+\Delta w}-e^w}{\Delta w}}$$

$\Big\{$ (*) $\Delta z = e^{w+\Delta w} - e^w$; $\operatorname{Ln}(z+\Delta z) = \operatorname{Ln}(e^{w+\Delta w}) = w+\Delta w$, and $\operatorname{Ln} z = \operatorname{Ln} e^w = w \Rightarrow \operatorname{Ln}(z+\Delta z) - \operatorname{Ln} z = w+\Delta w - w = \Delta w$.

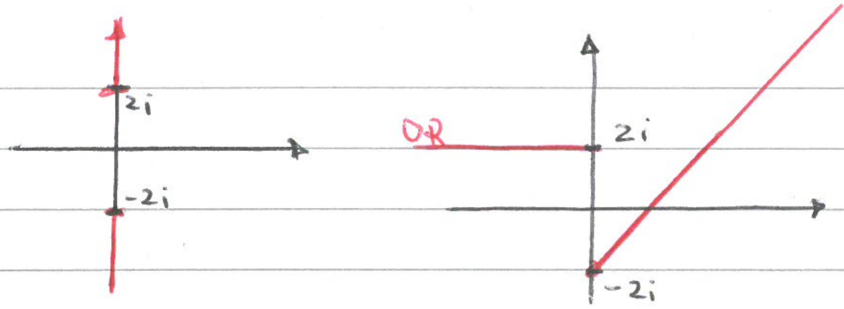
Passing to the limit $\Delta z \to 0$ we get:

$$\frac{dw}{dz} = \frac{1}{e^w} \Rightarrow \frac{d}{dz}(\operatorname{Ln} z) = \frac{1}{z} \qquad \blacksquare$$

<u>Ex 17.3</u>: Study the function $w = \operatorname{Ln}(z^2+4)$

<u>Sol</u>: $w = \operatorname{Ln}(z^2+4) = \operatorname{Ln}(z+2i)(z-2i) \underset{\text{Prop 15.1 1)}}{=} \operatorname{Ln}(z+2i) + \operatorname{Ln}(z-2i)$

$\Rightarrow$ w has 2 branch points $z=2i$ and $z=-2i$. Therefore, w is analytic in the cut-plane where we cut along 2 rays that start at $z=\pm 2i$ (including $\pm 2i$):



The derivative of this function in the cut-plane is computed as a composition function using:

By the Chain Rule (Prop 6.1 6)): $\left(\mathrm{Ln}(z^2+4)\right)' = \frac{2z}{z^2+4}$

Now we define the main branch of the function $\mathrm{Ln}\, z$.

Def 15.2: The branch of $w = \mathrm{Ln}\, z$ when $k=0$, $-\pi < \varphi < \pi$ is called the main branch of the logarithm: $w_0 = \mathrm{Ln}\, z = \ln r + i\varphi$

Having the logarithm, similarly to the way that we have defined the trigonometric and hyperbolic functions using the exponential function, we define the inverse of the trigonometric and hyperbolic functions as follows:

Def 15.3: 1) $\arcsin z$ is $w \in \mathbb{C}$ such that $\sin w = z$

2) $\arccos z$ is $w \in \mathbb{C}$ such that $\cos w = z$

3) $\mathrm{arctg}\, z$ is $w \in \mathbb{C}$ such that $\mathrm{tg}\, w = z$

From 1): $z = \sin w$. By Euler's formula $z = \frac{e^{iw} - e^{-iw}}{2i}$

Eliminating w we obtain: Denote $e^{iw} = t \Rightarrow z = \frac{t - \frac{1}{t}}{2i}$

$\Leftrightarrow t^2 - 2izt - 1 = 0 \Rightarrow t = iz \pm \sqrt{1-z^2}$ or $e^{iw} = iz \pm \sqrt{1-z^2}$, when there are two branches of (each) square root. Thus:

$$\arcsin z = w = \frac{1}{i}\mathrm{Ln}\left(iz \pm \sqrt{1-z^2}\right) = -i\,\mathrm{Ln}\left(iz \pm \sqrt{1-z^2}\right)$$

In the same way:

EXERCISE: $\arccos z = -i\,\mathrm{Ln}\left(z \pm i\sqrt{1-z^2}\right)$

$$\mathrm{arctg}\, z = \frac{1}{2i}\mathrm{Ln}\left(\frac{1+iz}{1-iz}\right)$$

Ex 17.4: Compute: 1) $\arcsin 2$ 2) $\arcsin i$

Sol: 1) $\arcsin 2 = -i\,\mathrm{Ln}\left(2i \pm \sqrt{1-4}\right) = -i\,\mathrm{Ln}\left((2 \pm \sqrt{3})i\right)$

$= -i\left(\ln(2 \pm \sqrt{3}) + i\left(\frac{\pi}{2} + 2\pi k\right)\right) = \frac{\pi}{2} + 2\pi k - i\ln(2 \pm \sqrt{3})$

If, for example, $k=1$ $\arcsin 2$ attains two values:

$$\arcsin 2 \approx \frac{5}{2}\pi \pm i \cdot 1.32$$

2) $\arcsin i = -i\,\mathrm{Ln}(-1 \pm \sqrt{2})$. Let us compute each of the logarithms:

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$Ln(-1+\sqrt{2}) = \ln(\sqrt{2}-1) + 2\pi k i$

$Ln(-1-\sqrt{2}) = \ln(\sqrt{2}+1) + (2k+1)\pi i$

$\Rightarrow \arcsin i = 2\pi k - i\ln(\sqrt{2}-1)$, $k \in \mathbb{Z}$ $(k = 0, \pm 1, \pm 2, \dots)$

$\arcsin i = \pi(2k+1) - i\ln(\sqrt{2}+1)$, $k \in \mathbb{Z}$

Rmk: ~~In~~ a domains where the corresponding branches of the square root and of the logarithm are single-valued, the inverse trigonometric functions are analytic and

$$(\arcsin z)' = \frac{1}{\sqrt{1-z^2}} \qquad (\arccos z)' = -\frac{1}{\sqrt{1-z^2}}$$

$$(\operatorname{arctg} z)' = \frac{1}{1+z^2}$$

Lectures 20+21.

Now we introduce isolated singularities and discuss their classification. Then we define the residue of a function and relate calculation of residues of certain singularities to zeros of analytic functions.

<u>Def 16.1</u>: If f is analytic on a punctured disc

$$D' = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$$

but it is not analytic at z_0 (or not defined at z_0), then z_0 is called an isolated singularity of f.

In other words, z_0 is an isolated singularity if there exists a neighborhood of z_0 such that f is analytic at every point in this neighborhood except at z_0. If z_0 is an isolated singularity of f, then there exists an annulus $0 < |z - z_0| < r$ (sufficiently small) such that f is analytic there and hence the Laurent series expansion of f exists on this annulus.

By Thm 14.2 (Laurent series) Such f has a Laurent series expansion valid on D'

$$(*) \quad f(z) = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = \dots \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z - z_0} + a_0 + \dots$$

The $\sum_{n=1}^{\infty} b_n (z - z_0)^n$ part of the series is called the <u>principal part</u> of the Laurent series.

<u>Note</u>: $\sum_{n=0}^{\infty} a_n (z - z_0)^n \xrightarrow[z \to z_0]{} a_0$

Now we can see that there are 3 options for the principal part in $(*)$: there is no principal part; there is principal part and it has finitely many terms; or there is an infinite principal part. This observation leads to the following classification of isolated singularities.

<u>Three types of isolated singularities.</u>

<u>I. Removable singularity</u>: If $b_n = 0$ for all n, namely the principal part of the Laurent series is identically 0, then $z = z_0$ is called the removable singularity.

Namely, the Laurent series is an analytic function on the disc centered at z_0, including the point z_0. At every interior point of that disc the series converges to the function $f(z)$ and at z_0

($f(z)$ is not analytic at z_0) it converges to a_0.

Define: $\tilde{f}(z) = \begin{cases} f(z) & z \neq z_0 \\ a_0 & z = z_0 \end{cases}$. Then, $\tilde{f}(z)$ is the value of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on $|z-z_0|<r$ (including $z=z_0$) and $\tilde{f}(z)$ is analytic at the point $z=z_0$.

II Pole: If the principal part of the Laurent series $\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ has finite strictly positive number of non-zero terms, namely, if it has the form $\sum_{n=1}^{m} b_n(z-z_0)^{-n}$ with $b_m \neq 0$, then z_0 is called a pole of order m. A pole of order $m=1$ is called a simple pole.

III Essential singularity: If $b_n \neq 0$ for infinitely many n, then z_0 is called an essential singularity.

Def 16.2: The coefficient b_1 of $\frac{1}{z-z_0}$ term is called the residue of the singularity at $z=z_0$, denoted by $\operatorname{Res}_{z=z_0} f(z) = \operatorname{Res}(f, z_0)$

First, let us study the case of removable singularity. The following proposition is a useful criterion to check for removable singularity.

Prop 16.1: z_0 is a removable singularity of $f(z)$ if and only if $\lim_{z\to z_0} (z-z_0)f(z) = 0$.

Pf: $\boxed{\Rightarrow}$ Suppose z_0 is removable and g is analytic on the open disc centered at z_0 of radius R: $D = \{z \in \mathbb{C} : 0 \leq |z-z_0| < R\}$, such that $f=g$ for $z \neq z_0$. Then, since g is continuous at z_0 (Prop 6.5) and we have

$$\lim_{z\to z_0} (z-z_0)f(z) = \lim_{z\to z_0} (z-z_0)g(z) = \lim_{z\to z_0}(z-z_0) \lim_{z\to z_0} g(z) = 0 \cdot g(z_0) = 0$$

$\boxed{\Leftarrow}$ Suppose that $\lim_{z\to z_0} (z-z_0)f(z) = 0$ and f is analytic on the punctured disc $D' = \{z \in \mathbb{C} : 0 < |z-z_0| < R\}$. Define:

$g(z) = \begin{cases} (z-z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$. g is analytic for $z \neq z_0$ and it is differentiable at z_0:

$$g'(z_0) = \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{(z-z_0)^2 f(z)}{z-z_0} = \lim_{z\to z_0} (z-z_0)f(z) = 0$$

$\Rightarrow$ g is analytic on D with $g(z_0) = g'(z_0) = 0$ $\Rightarrow$ g has power series expansion $g(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ with $a_0 = a_1 = 0$ $\Rightarrow$ we can factor $(z-z_0)^2$ from the series, so

$$g(z) = (z-z_0)^2 \sum_{n=0}^{\infty} a_{n+2}(z-z_0)^n = (z-z_0)^2 f(z)$$

Therefore, for $z \neq z_0$ we have: $f(z) = \sum_{n=0}^{\infty} a_{n+2}(z-z_0)^n$ and this series defines an analytic function on D ∎

The other types of singularities can also be recognised by the behavior of f near it.

1) If z_0 is a pole, then $\lim_{z \to z_0} f(z) = \infty$:
If z_0 is a pole of order m, then using the Laurent series expansion we get

$$f(z) = b_m(z-z_0)^{-m} + \dots + b_1(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \dots$$
$$= \frac{1}{(z-z_0)^m}\left[b_m + b_{m-1}(z-z_0) + \dots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots\right]$$

Therefore, since $\lim_{z \to z_0} \frac{1}{(z-z_0)^m} = \infty$ and $\lim_{z \to z_0}[b_m + \dots] = b_m$, we conclude that $\lim_{z \to z_0} f(z) = \infty$.

2) If z_0 is an essential singularity, then $\lim_{z \to z_0} f(z)$ does not exist – highly non-obvious!

In fact, more is true in the case of an essential singularity:

<u>Thm</u> (<u>Great Picard Theorem</u>) If an analytic function f has an essential singularity at z_0, then on any punctured neighborhood of z_0 f takes on all possible values in $\mathbb{C}$ infinitely often with at most one exception.

In other words, if z_0 is an essential singularity of the function f, then the equation $f(z) = c$ for any (finite) c except possibly for one value, has infinitely many solutions that tend to z_0.

Similarly to Prop 16.1, we have the following proposition which provides a method for recognising the order of a pole and computing the corresponding residue.

<u>Prop 16.2</u>: $z = z_0$ is a pole of order $m > 0$ of $f \iff f$ can be expressed as $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic at a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$.

Moreover, $\operatorname{Res}(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

<u>Pf</u>: $\boxed{\Rightarrow}$ If z_0 is a pole of order m, then as we have seen

$$f(z) = \frac{1}{(z-z_0)^m}\left[b_m + b_{m-1}(z-z_0) + \dots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots\right]$$

Denote: $\varphi(z) = b_m + b_{m-1}(z-z_0) + \dots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots$
Then, φ is analytic at a neighborhood of z_0 (this is its Taylor series expansion!) including z_0 and $\varphi(z_0) = b_m \neq 0$.

$\boxed{\Leftarrow}$ Assume $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$, where φ is analytic in a neighborhood of z_0 (including z_0) and $\varphi(z_0) \neq 0$. Since φ is analytic there exists Taylor series expansion in some neighborhood of z_0, given by

$$\varphi(z) = \varphi(z_0) + \frac{\varphi'(z_0)}{1!}(z-z_0) + \dots + \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^m + \dots$$

$\Rightarrow$ f can be written in terms of series expansion as

$$f(z) = \frac{\varphi(z_0)}{(z-z_0)^m} + \frac{\varphi'(z_0)}{(z-z_0)^{m-1}} + \dots + \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}\frac{1}{z-z_0} + \frac{\varphi^{(m)}(z_0)}{m!} + \dots$$

This is a Laurent series with finitely many terms in its principal part. Since $\varphi(z_0) \neq 0$, f has a pole of order m at z_0. The residue is the coefficient of $\frac{1}{z-z_0}$, namely $\frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$ $\blacksquare$

Let us see examples of these singularities.

Ex 18.1: Let $f(z) = \frac{1}{(z-1)(z-2)}$. Then f has isolated singularities at $z=1$ and $z=2$ {f is a rational function $\Rightarrow$ analytic everywhere except at the zeros of the denominator}

CHECK: The Laurent series about the point $z_0 = 1$ is given by

$$f(z) = -\frac{1}{z-1} + \sum_{n=0}^{\infty} -(z-1)^n$$

The Laurent series at $z_0 = 2$: Observe that

$$f(z) = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2}\cdot\frac{1}{1+(z-2)} = \frac{1}{z-2}\sum_{n=0}^{\infty}(-1)^n(z-2)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n(z-2)^{n-1} = \frac{1}{z-2} + \sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^{n-1}$$

The series is valid on a punctured disc centered at $z_0=2$ of radius 1: $D' = \{z \in \mathbb{C} : 0 < |z-2| < 1\}$

Claim: $z=1, z=2$ are simple poles.

Pf: $z=2$: Rewrite $f(z)$ in the form $f(z) = \frac{\varphi(z)}{z-1}$, where $\varphi(z) = \frac{1}{z-1}$.
Then $\varphi(z)$ is analytic in a neighborhood of $z=2$ (including $z=2$) and $\varphi(2) = 1 \neq 0$, thus, by Prop 16.2 $z=2$ is a simple pole.

$z=1$: In the same way: $f(z) = \frac{\tilde{\varphi}(z)}{z-1}$, $\tilde{\varphi}(z) = \frac{1}{z-2}$, $\tilde{\varphi}$ is analytic in a neighborhood of $z=1$ (including $z=1$), and $\tilde{\varphi}(1) = -1 \neq 0$.
Thus, by Prop 16.2 $z=1$ is a simple pole.

By Prop 16.2: $\mathrm{Res}(f,2) = \frac{\varphi(2)}{0!} = 1$; $\mathrm{Res}(f,1) = \frac{\tilde{\varphi}(1)}{0!} = -1$.

Ex 18.2 Prove:

1) $z=3$ is a removable singularity of $f(z) = \frac{z^2-9}{z-3}$

2) $z=0$ is a removable singularity of $f(z) = \frac{\sin z}{z}$

Sol: 1) Since $\lim\limits_{z\to 3} \frac{(z-3)(z+3)}{(z-3)} = 6$ and $\lim\limits_{z\to 3} (z-3)\cdot\frac{(z-3)(z+3)}{z-3} = 0$

by Prop 16.1 $z=3$ is a removable singularity. Define:

$$\tilde{f}(z) = \begin{cases} \frac{z^2-9}{z-3} & z \neq 3 \\ 6 & z=3 \end{cases} \Rightarrow \tilde{f}(z) \text{ is entire (analytic on } \mathbb{C})$$

2) We prove this in two ways:

i) The Laurent series for f in $0<|z|<R$ is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$\Rightarrow$ the principal part does not exist $\Rightarrow$ $z=0$ is a removable singularity

ii) $\lim\limits_{z\to 0} z\cdot\frac{\sin z}{z} = \lim\limits_{z\to 0} \sin z = 0 \Rightarrow$ by Prop 16.1 $z=0$ is a removable singularity.

Ex 18.3: Classify the singularities of $f(z) = e^{\frac{1}{z}}$ and compute the residue of f at $z=0$.

Sol: Since $f(z)$ is analytic everywhere except for $z=0$, f has an isolated singularity at $z=0$. To determine the nature of this singularity, we write the Laurent series expansion:

$$e^{1/z} = e^{z^{-1}} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \dots$$

$\Rightarrow$ the principal part $\sum\limits_{n=1}^{\infty} b_n (z-z_0)^{-n}$ has infinitely many non-zero terms $\Rightarrow$ $z=0$ is an essential singularity.

By Picard's Great Theorem, f takes every value of $\mathbb{C}\setminus\{0\}$ infinitely often. The coefficient of the term $\frac{1}{z}$ in the Laurent series expansion is 1, therefore $\mathrm{Res}(e^{\frac{1}{z}}, 0) = b_1 = 1$.