Week 4. Lecture 10

We have studied the continuity and differentiability of complex functions. Our next subject – analytic functions.

Single-valued analytic functions.

Recall: If for any z corresponds one value $f(z)$, then we say that f is single-valued function.

Def 7.1: Single-valued function $f(z)$ is analytic at the point z_0 if there exists ε-neighborhood $D(z_0, \varepsilon)$ of z_0 so that f is differentiable at every $z \in D(z_0, \varepsilon)$

We have proved that if f is differentiable at z_0, then f is continuous at z_0, therefore if f is analytic at z_0, then f is continuous at z_0.

Let us note that we will also use names Holomorphic or Regular functions for analytic functions.

Def 7.2: A function that is analytic for every $z \in \mathbb{C}$ (on the whole plane) is called an entire function.

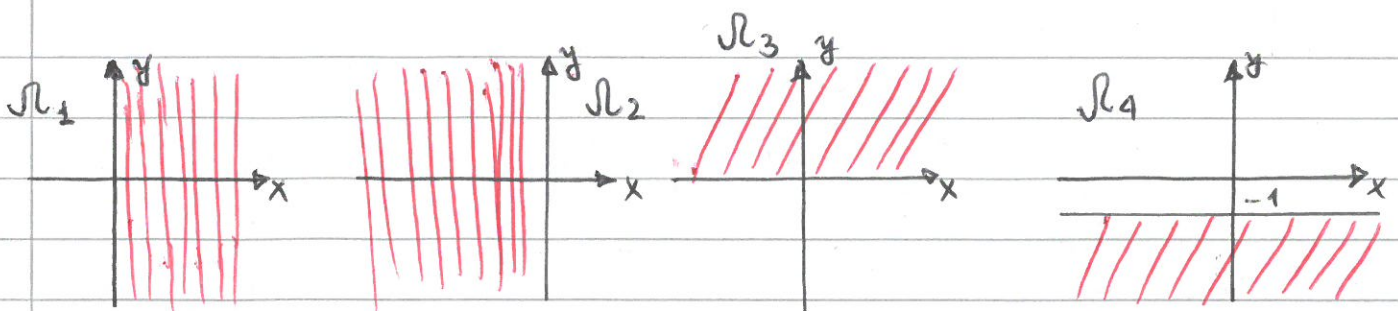Def 7.3: Single-valued function is analytic on a domain (open connected subset of the plane $\mathbb{C}$) Ω if it is differentiable at every $z \in \Omega$.

Note: From Def 7.1: a set on which the function is analytic needs to be open, however if we say that some function f is analytic in a closed set, we mean that there exists an open set that contains this closed set and the function is analytic on this open set.
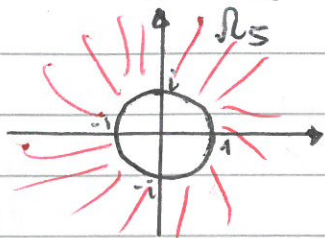
Ex 9.1: Polynomial $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ – single-valued function differentiable at every $z \in \mathbb{C}$ $\Rightarrow$ analytic on $\mathbb{C}$ $\Rightarrow$ entire.

Ex 9.2: Every rational function $f(z) = \dfrac{P_n(z)}{Q_m(z)}$ where $P_n(z)$, $Q_m(z)$ are polynomials (of deg n, m respectively) without common factors is analytic on $\mathbb{C}$ except for the roots of $Q_m(z)$, namely except for the points where $Q_m(z) = 0$.

For instance: $f(z) = \dfrac{z-1}{z^2 + iz} = \dfrac{z-1}{z(z+i)}$ is analytic everywhere except for $z = 0, -i$. For example, it is analytic on the following domains:
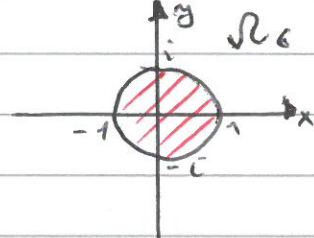
$\{z \in \mathbb{C}: \text{Re}\, z > 0\}$ $\{z \in \mathbb{C}: \text{Re}\, z < 0\}$ $\{z \in \mathbb{C}: \text{Im}\, z > 0\}$ $\{z \in \mathbb{C}: \text{Im}\, z < -1\}$

$\{z \in \mathbb{C}: |z| > 1\}$

But it is not <u>analytic</u> in Ω_6, since $0 \in \{z \in \mathbb{C}: |z| < 1\}$ and f is not defined at $z = 0$ $\{z \in \mathbb{C}: |z| < 1\}$

<u>Note</u>: f is analytic on a domain $\Omega \iff u$ and v are differentiable on Ω and CR equations hold.

<u>Exercise</u> (Ex 9.3) The function $f(z) = e^x \cos y + i e^x \sin y = e^z$ is entire.

In the next example: a function for which CR equations hold on the whole plane, but u and v are <u>not</u> differentiable on the whole plane, thus the function is <u>not</u> entire.

<u>Ex 9.4</u>: $f(z) = \begin{cases} e^{-1/z^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$ defined everywhere in $\mathbb{C}$

<u>CHECK</u>: CR equations hold for every $z \in \mathbb{C}$

<u>Claim</u>: $f(z)$ is not continuous at $z = 0$ ($\implies$ <u>not</u> differentiable)

<u>Pf</u>: compute $\lim\limits_{z \to 0} f(z)$ on the path $y = x$:

$$\lim_{\substack{x \to 0 \\ y = x}} e^{-\frac{1}{(x+iy)^4}} = \lim_{x \to 0} e^{-\frac{1}{(1+i)^4 x^4}} \underset{\uparrow}{=} \lim_{x \to 0} e^{\frac{1}{4x^4}} = +\infty$$

$$(1+i)^4 \underset{\uparrow}{=} (\sqrt{2})^4 \left(\cos\left(\tfrac{\pi}{4} 4\right) + i \sin\left(\tfrac{\pi}{4} 4\right)\right) = -4$$

De Moivre

Next proposition is analogous to propositions about the limits, continuity, and differentiability.

<u>Prop 7.1</u>: If f, g are analytic functions on a domain Ω, then also $f+g$ and $f \cdot g$ are analytic on Ω. If $g(z) \neq 0$ on Ω, then $\frac{f}{g}$ is analytic on Ω. The composition $(g \circ f)$ of two analytic functions on Ω is an analytic function on Ω.

Pf: EXERCISE.

Ex 9.5: Prove: if $f(z)$ is analytic and real on a domain Ω, then f is a constant function.

Pf: f is analytic and real on Ω $\Rightarrow$ $f(x,y) = u(x,y) + i \cdot 0$, namely $v(x,y) = 0$; CR equations hold for every $z \in \Omega$ { from analyticity on Ω}: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$ $\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ $\Rightarrow$ $u(x,y) = c \in \mathbb{R}$ $\Rightarrow$ $f(x,y) = c$, namely f is a constant function. ▨

Prop 7.2: If f is analytic on a domain Ω and $f'(z) = 0$ for all $z \in \Omega$, then f is constant on Ω.

Rmk: Prop 7.2 is false if Ω is not connected — f can take different constants (constant values) on each connected component of Ω.

Pf: The version of this proposition for real functions is proved using the Mean Value Theorem, that asserts that under appropriate continuity assumption for $f: (a,b) \to \mathbb{R}$ there exists $c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$. So, if $f'(x) = 0$ for all $x \in (a,b)$, then $f(b) = f(a)$. The problem in the complex case is that there is no analogous idea of "between a and b" in $\mathbb{C}$, since there are many routes from a to b. However, we can use the Mean Value Theorem for $\text{Re } f$ and $\text{Im } f$ as follows:

Since $f'(z) = 0$ for all $z \in \Omega$ by CR equations
$$0 = f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{for all } z \in \Omega$$
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{for all } z \in \Omega.$$

Claim: u is a constant function.

Pf: We have $u: \Omega \to \mathbb{R}$, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$ for all $(x,y) \in \Omega$. Pick $(x_0, y_0), (x_1, y_1) \in \Omega$. Ω is connected, thus by Def 4.2 any 2 points in Ω can be connected by a path that is entirely in Ω. We join (x_0, y_0) and (x_1, y_1) by a path comprised of pieces parallel to x-axis or y-axis [One needs to show that such path exists!] Since $\frac{\partial u}{\partial x} = 0$ by the Mean Value Theorem, u is constant on the horizontal pieces. In the same way

since $\frac{\partial u}{\partial y} = 0$ u is constant on vertical pieces. Thus, u is constant on Ω. ⬛

By a similar argument we obtain that v is constant on Ω, therefore $f = u + iv$ is constant on Ω. ⬛

Now we are going to study analiticity of multi-valued functions - here we will see the big difference between the complex and real functions.

<u>Analyticity of multi-valued functions</u>

We have seen that analytic function is always continuous, thus multi-valued functions are not analytic as is. The question is: can we choose a <u>branch</u> of such function that is single-valued and analytic?

<u>Def 8.1</u>: We say that on a domain Ω there is a single-valued branch of multi-valued function if for every point of Ω we can choose one of the values of a function such that the resulting single-valued function is continuous in Ω.

In other words: to choose single-valued branch means to choose the domain of definition of a function such that when a point z moves along some closed continuous curve that is contained in the domain of definition and goes back to the starting point we get the same value of the function.

Let us demonstrate how we do that on the following example.

<u>Ex 10.1</u>: Consider double-valued function $f(z) = \sqrt{z - z_0}$.

$f(z)$ is defined for all $z \in \mathbb{C}$. Write:

$$z - z_0 = r(\cos\theta + i\sin\theta), \quad r = |z - z_0|, \quad \theta = \arg(z - z_0)$$

At every $z \neq z_0$ $f(z)$ attains 2 values ($\Rightarrow$ double-valued!)

$$w_0 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$w_1 = \sqrt{r}\left(\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right) = -\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = -w_0$$

Let us follow the value w_0 when z is on the closed continuous curve L that encircles z_0 counterclockwise (anti-clockwise; the positive direction)
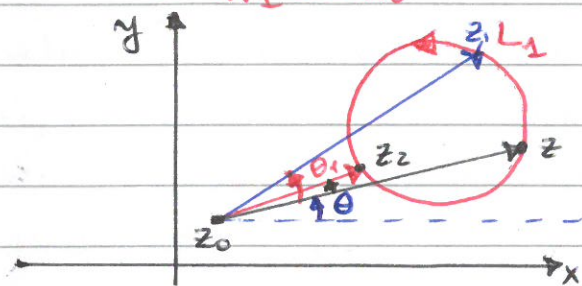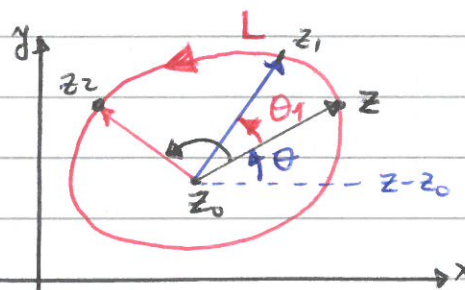
If z moves to z_1, the argument of $z_1 - z_0$ will be

$$\arg(z_1 - z_0) = \theta + \theta_1$$

If continue to move along L, z will eventually return to the starting point: the modulus $r = |z - z_0|$ will be the same, but the argument will increase by 2π (since we completed the circle) and w_0 will attain the new value w_1.

$$w_{new} = \sqrt{r}\left(\cos\left(\frac{\theta + 2\pi}{2}\right) + i\sin\left(\frac{\theta + 2\pi}{2}\right)\right) = \sqrt{r}\left(\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right)$$

$$= w_1 = -w_0$$

Let us check what happens to w_0 is we move along a closed continuous curve that does not encircle the point z_0.

From the picture above that the value of the modulus and of the argument after completing the full circle will return to its starting value — the same, with no change. This is independent of the curve L_1 (the only important thing that the curve does not encircle the point z_0)

<u>Cor 8.1</u>: On a domain Ω that contains curves that encircle z_0 it is impossible to choose a continuous branch of the function $\sqrt{z - z_0}$.

Lectures 11 + 12.

We have studied the double-valued function $f(z) = \sqrt{z-z_0}$

Cor 8.1: On a domain Ω that contains curves that encircle z_0 it is impossible to choose a continuous branch of $\sqrt{z-z_0}$

For example: On $\frac{1}{2} < |z-z_0| < 2$ – impossible to choose single-valued branch of $\sqrt{z-z_0}$.
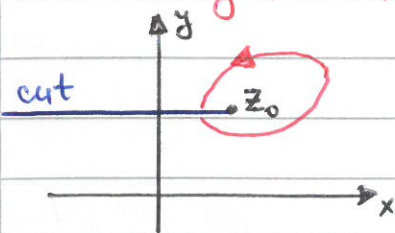
On $|z-z_0| > 1$ impossible to choose single-valued branch of $\sqrt{z-z_0}$

We have also seen that if the curve does not encircle z_0, then the value of $\sqrt{z-z_0}$ stays the same. So, we have another corollary:
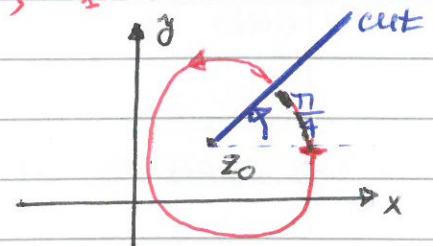
Cor 8.2: It is possible to choose a single-valued branch of the function $\sqrt{z-z_0}$ on a domain Ω if none of the closed continuous curves in Ω encircle z_0.

Intuitively: if we would able to "cut" the plane (in some way) up to z_0, including z_0, then there will be no closed curve that encircles z_0. Indeed, the biggest domain on which none of the closed continuous curves encircle z_0 is the cutted plane, where we cut as follows:

The "largest" such domain: Cut-plane – $\mathbb{C}$ without the line $\{z = x+iy \mid x \le x_0\}$ [including z_0]

Cut-plane: two single-valued branches
$$w_0 = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right), \quad w_1 = -\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

Note: can cut along any ray starting at z_0

The cut will prevent from any closed curve to encircle z_0. In case like on the picture above (ray creates an angle $\frac{\pi}{4}$ with the positive – anticlockwise – direction with x-axis) we have:

For a ray with $\theta = \frac{\pi}{4}$: 2 continuous branches of $\sqrt{z-z_0}$:
$$w = \pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \qquad \frac{\pi}{4} < \theta \le \frac{9\pi}{4}$$

Def 8.2: The point z_0 is called a branch point: for complex multi-valued function f if the value of $f(z)$ does not return to

its initial value as a closed curve around the point is traced (starting from some arbitrary point on the curve), in such a way that f varies continuously as the path is traced.

<u>Important</u>. What matters here is the <u>local</u> behavior of the function f near z_0. What may happen on paths that are some distance away from z_0 is not relevant. To be more precise:

The behavior must occur for all the curves that enclose the point and are sufficiently close to it.

In Ex 10.1: $z = z_0$ is the <u>Branch Point</u>

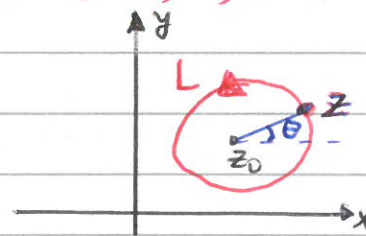Let us consider a more general case.

<u>Ex 10.2</u>: Consider $w = \sqrt[n]{z - z_0}$. Take $z - z_0 = r(\cos\theta + i\sin\theta)$. Then

$$w = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \quad k = 0, 1, 2, \ldots, n-1$$

Fix $0 \le k_0 \le n-1$.

As before: we let z to be on the closed continuous curve L that encloses z_0.

When z will complete the full circle (anticlockwise) the modulus (r) will return to the same value, the argument however will increase by 2π and w will attain a new value:

After full circle around z_0:

$$W_{new} = \sqrt[n]{r}\left(\cos\left(\frac{(\theta + 2\pi) + 2\pi k_0}{n}\right) + i\sin\left(\frac{(\theta + 2\pi) + 2\pi k_0}{n}\right)\right)$$

$$= \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k_0}{n}\right) + i\sin\left(\frac{\theta + 2\pi k_0}{n}\right)\right)\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)$$

$$= W_{k_0}\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right) = W_{k_0+1}$$

Namely, the new value of w is equal to the previous one times $\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)$

CHECK: $\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = 1$

This is independent of the curve that encloses z_0! If z will run on a curve that does not encircle z_0, then, as before, the value W_{k_0} will not change once z returns to the starting point.

<u>Conclusion</u>: If z is on the curve that encloses z_0 once (anticlockwise), then the value $\sqrt[n]{z - z_0} = W_{k_0}$ will change to the new value W_{k_0+1}. If z is on the curve that does not enclose z_0
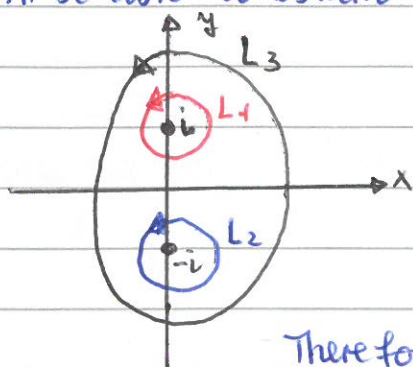
then the value of the function stays the same.

$\Rightarrow$ $z = z_0$ is the **branch point** of the function $f(z) = \sqrt[n]{z - z_0}$.

Ex 10.3: Find the branch points of the function $f(z) = \sqrt{1 + z^2}$

Sol: $\sqrt{1 + z^2} = \sqrt{z - i} \cdot \sqrt{z + i}$, therefore $z = \pm i$ are the branch points of $\sqrt{z - i}$, $\sqrt{z + i}$ $\Rightarrow$ branch points of $\sqrt{1 + z^2}$.

Let us see on the picture what happens and where to cut so we will be able to obtain a continuous single-valued branch of $f(z)$.
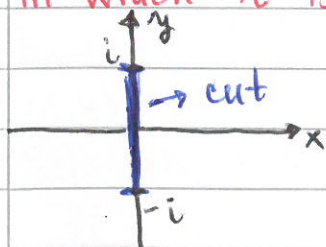

If z is on L_1 (around $z = i$): When z returns to the initial point $\sqrt{z + i}$ returns to the same value since L_1 does not enclose $z = -i$ while $\sqrt{z - i}$ attains new value - changes the sign of the previous one. Therefore, $f(z)$ also changes the sign. In the same way: if z is on L_2, then upon return to the initial point $\sqrt{z + i}$ will change the value - change the sign, $\sqrt{z - i}$ will return to the same value, and the function will change the sign. If z is on L_3 that encloses both $z_0 = i$ and $z_0 = -i$, then $\sqrt{z - i}$ and $\sqrt{z + i}$ will change the sign, but their product - the original function - will not!

$\Rightarrow$ We can choose a continuous branch on a domain Ω if none of the closed curves in Ω enclose i and not $-i$ (or $-i$ and not i), but it can enclose both. Namely, if we cut the plane along the interval connecting i and $-i$, we will get a domain in which it is possible to choose a continuous branch.

Def 8.3: Single-valued branch is called an analytic function on a domain Ω if it is differentiable at every $z \in \Omega$.

Ex 10.4: Prove that $w = \sqrt[n]{z}$ is analytic on the domain
$$\Omega = \{ z \in \mathbb{C} : \ 0 < \arg z < \frac{2\pi}{n} \}.$$

Pf: We have already seen that on the cut-plane with the cut

along any ray starting at $z_0 = 0$ (including $z_0 = 0$) every branch of this function is single-valued and continuous. Let us prove that it is differentiable and find the derivative.

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{(f(z))^n - (f(z_0))^n}{z - z_0} \cdot \frac{1}{(f(z))^{n-1} + (f(z))^{n-2} f(z_0) + \dots + (f(z_0))^{n-1}}$$

$$= \frac{(\sqrt[n]{z})^n - (\sqrt[n]{z_0})^n}{z - z_0} \cdot \frac{1}{(\sqrt[n]{z})^{n-1} + (\sqrt[n]{z})^{n-2} \sqrt[n]{z_0} + \dots + (\sqrt[n]{z_0})^{n-1}}$$

$$= \frac{1}{(\sqrt[n]{z})^{n-1} + (\sqrt[n]{z})^{n-2} \sqrt[n]{z_0} + \dots + (\sqrt[n]{z_0})^{n-1}}$$

Now we pass to the limit as z tends to z_0 and obtain

$$\Rightarrow \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{n(\sqrt[n]{z_0})^{n-1}}$$

$$\Rightarrow f \text{ is differentiable and } (\sqrt[n]{z})' = \frac{1}{n(\sqrt[n]{z})^{n-1}}$$

In many problems in hydrodynamics, the elasticity theory and other fields there is often a need to recreate an analytic function from its real part. Now we will find the set of functions for which it is possible. Our subject is:

Harmonic functions.

Def 9.1: The real function $u(x,y)$ is called harmonic on a domain Ω if it is continuous on Ω with continuous partial derivatives of the first and second order $\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x} \}$, and the following equation, called the Laplace equation holds:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Ex 11.1: The function $u(x,y) = x^2 - y^2$ is harmonic for every $(x,y) \in \mathbb{R}^2$.

Sol: $u(x,y)$ is defined everywhere, and we have:

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2y \quad \frac{\partial^2 u}{\partial x^2} = 2 \quad \frac{\partial^2 u}{\partial y^2} = -2 \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 0$$

All partial derivatives are continuous everywhere, and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + (-2) = 0$$

Ex 11.2 The function $u(x,y) = e^{xy}$ is not harmonic function on $\mathbb{R}^2$.

Pf: $u(x,y)$ is defined everywhere on $\mathbb{R}^2$

$$\frac{\partial u}{\partial x} = ye^{xy} \qquad \frac{\partial u}{\partial y} = xe^{xy} \qquad \frac{\partial^2 u}{\partial x^2} = y^2 e^{xy} \qquad \frac{\partial^2 u}{\partial y^2} = x^2 e^{xy}$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{xy} + yxe^{xy} \qquad \frac{\partial^2 u}{\partial y \partial x} = e^{xy} + xye^{xy}$$

All partial derivatives are defined and continuous for all $(x,y) \in \mathbb{R}^2$. However

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y^2 e^{xy} + x^2 e^{xy} = e^{xy}(x^2 + y^2) \neq 0 \text{ except for}$$

$(0,0) \Rightarrow u(x,y)$ is __not__ harmonic function on $\mathbb{R}^2$.

__Prop 9.1__: If the function $f(z) = u(x,y) + iv(x,y)$ is analytic on a domain Ω, then $u(x,y)$ and $v(x,y)$ are harmonic on Ω.

__Pf__: By CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (since f is analytic). Let us differentiate the first equation with respect to x and ~~the~~ second with respect to y:

$$\circledast \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} \; ; \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Since f is analytic u and v are continuously differentiable, thus from theory of real functions, we know that

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$$

Therefore, if we add the equations $\circledast$, we obtain:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$\Rightarrow u$ is harmonic function on Ω.

If we differentiate now $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ with respect to y and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ with respect to x, and use $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ we will obtain that v is also harmonic function on Ω. ▨

This proposition implies that in order to construct an analytic function we cannot choose the real and imaginary parts in an arbitrary way. The functions that we choose must be harmonic and, moreover, the CR equations must hold.

__Def 9.2__: If $u(x,y)$ and $v(x,y)$ are harmonic functions on a domain Ω such that CR equations hold $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$, then the function $v(x,y)$ is called the __harmonic conjugate__ of the function $u(x,y)$ on a domain Ω.

__Prop 9.2__: The function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain $\Omega \iff v(x,y)$ is the harmonic conjugate of $u(x,y)$.

__Pf__: $\boxed{\Rightarrow}$ Assume f is analytic in a domain Ω. By Prop 9.1

$u(x,y)$ and $v(x,y)$ are harmonic functions on Ω. Since f is analytic CR equations hold, thus by Def 9.2 $v(x,y)$ is the harmonic conjugate of $u(x,y)$.

$\boxed{\Longleftarrow}$ If $u(x,y)$ and $v(x,y)$ are harmonic functions on a domain Ω, then u and v are differentiable at all $z \in \Omega$. Since $v(x,y)$ is the harmonic conjugate of $u(x,y)$, by Def 9.2 - CR equations hold, thus Thm 6.1 asserts that f is analytic in a domain Ω. ▨

Thus, to conclude that we can reconstruct an analytic function from its real part we need to know whether it is always possible to find a harmonic conjugate to a given harmonic function. The answer is yes — it is given by the following proposition which we state without the proof.

<u>Prop 9.3</u>: For any harmonic function $u(x,y)$ in a domain Ω there exists a function $v(x,y)$ in Ω which is the harmonic conjugate of $u(x,y)$.

The proof is by construction: one explicitly constructs the function $v(x,y)$. Since the proof involves somewhat complicated real analysis, we omit it.

<u>Ex 11.3</u>: Find an analytic function $f(z) = u(x,y) + i v(x,y)$ such that $u(x,y) = \mathrm{Re}\, f(z) = \mathrm{arctg}\, \frac{y}{x}$.

<u>Sol</u>: Let us compute the partial derivatives and then use CR equations: Recall: $(\mathrm{arctg}\, x)' = \frac{1}{1+x^2}$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2} \qquad \frac{\partial u}{\partial y} = \frac{x}{x^2+y^2} \underset{\underset{\text{CR equations}}{\downarrow}}{\Longrightarrow} \frac{\partial v}{\partial y} = \frac{-y}{x^2+y^2}, \quad \frac{\partial v}{\partial x} = \frac{-x}{x^2+y^2}$$

To obtain $v(x,y)$ we integrate $\frac{\partial v}{\partial x}$ with respect to x (or $\frac{\partial v}{\partial y}$ with respect to y)

ⓐ $v(x,y) = \int \frac{\partial v}{\partial x} dx = \int \frac{-x}{x^2+y^2} dx = -\frac{1}{2} \ln(x^2+y^2) + c(y)$

Since the integral is with respect to x, the constant may depend on y (only!) To find $c(y)$ we differentiate $v(x,y)$ in ⓐ with respect to y and compare it to $\frac{\partial v}{\partial y}$ obtained from CR equations

$\frac{\partial v}{\partial y} = \frac{-y}{x^2+y^2} + c'(y) = \frac{-y}{x^2+y^2} \Rightarrow c'(y) = 0 \Rightarrow c = \text{constant} \in \mathbb{R}$

$\Rightarrow v(x,y) = -\frac{1}{2} \ln(x^2+y^2) + c$ and $f(z) = \mathrm{arctg}\, \frac{y}{x} - \frac{i}{2}[\ln(x^2+y^2) + 2c]$