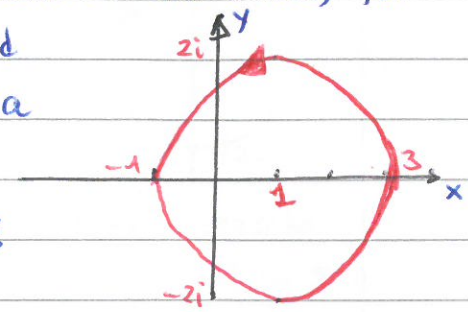
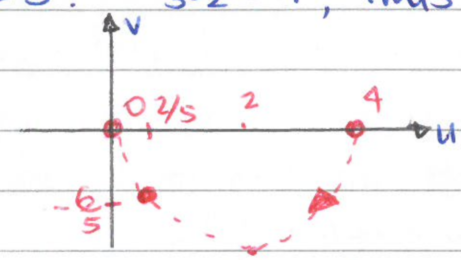
Problem Set 6 - Solutions.

1) a) We have seen that the boundary is mapped to the boundary. First, let us check that the boundary $|z-1|=2$ is mapped to $|w-2|=2$. We have studied that the Möbius transformation maps lines and circles into lines and circles. Either of them is determined by 3 points. We take, for instance, $z=-1$, $z=1+2i$, $z=3$, and check whether the image is a line or a circle. $w=\frac{z+1}{z-2} \Rightarrow z=-1: \frac{-1+1}{-1-2}=0$



$z=1+2i: \quad \frac{1+2i+1}{1+2i-2}=\frac{2+2i}{-1+2i}=\frac{2}{5}-i\frac{6}{5}$

$z=3: \quad \frac{3+1}{3-2}=4$, thus we get the following picture:



$\Rightarrow w_1, w_2, w_3$ are on a circle (rather than on a line) - the circle "starts" at $w=4$, "ends" at $w=0$ with the point $\frac{2}{5}-i\frac{6}{5}$ lying on it. Let us check that $\frac{2}{5}-i\frac{6}{5}$ is on $|w-2|=2$:

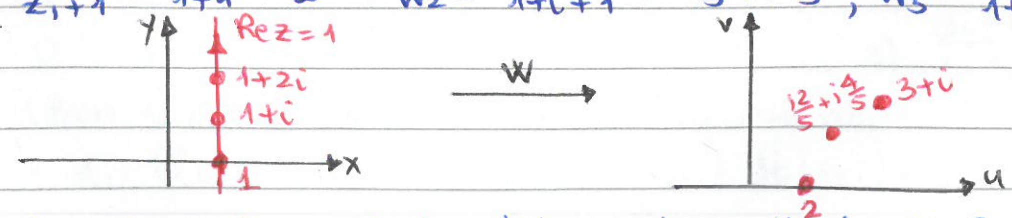
$$\left|\frac{2}{5}-i\frac{6}{5}-2\right|=\left|-\frac{8}{5}-i\frac{6}{5}\right|=\frac{\sqrt{64+36}}{5}=\frac{10}{5}=2$$

The boundary is mapped to the boundary $\Rightarrow |z-1|=2 \to |w-2|=2$

To check where to the interior of $|z-1|<2$ is mapped, let us take a point from the interior, for instance, $z_0=1$ and check where to z_0 is mapped: $\frac{1+1}{1-2}=-2 \to$ outside of $|w-2|\leq 2$, namely inside $|w-2|>2$ (the exterior of $|w-2|=2$)

b) As before, first we check what is the image of the boundary $\{z: \operatorname{Re} z=1\}$. Take 3 points: $z_1=1$, $z_2=1+i$, $z_3=1+2i$. Then

$w_1=\frac{4z_1}{z_1+1}=\frac{4}{1+1}=2 \qquad w_2=\frac{4(1+i)}{1+i+1}=\frac{12}{5}+i\frac{4}{5}, \quad w_3=\frac{4(1+2i)}{1+2i+1}=3+i$



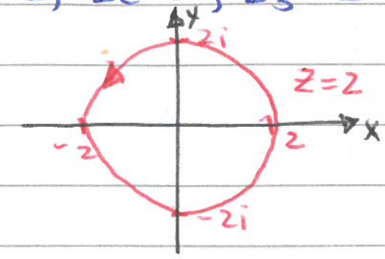
$\Rightarrow$ the boundary is a circle. Let us check that $w_2 \in |w-3|=1$

$$\left|\frac{12}{5}+i\frac{4}{5}-3\right|=\left|-\frac{3}{5}+i\frac{4}{5}\right|=\frac{\sqrt{9+16}}{5}=\frac{5}{5}=1.$$

To see what is the image of $\operatorname{Re} z<1$ we take some point from there, for example $z=0$. Then

$0 \to \frac{4\cdot 0}{0+1} = 0 \Rightarrow$ outside of $|w-3|=1$

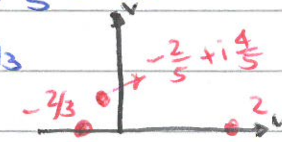
c) Here we have 2 boundaries: $|z|=2$ and $|z|=1$. First, let us show that the circle $|z|=2$ is mapped on the circle $|w-\frac{2}{3}| = \frac{4}{3}$. Take 3 points: $z_1=-2$, $z_2=2i$, $z_3=2$ (all on the boundary)



$z_1 \to \frac{2}{-2-1} = -\frac{2}{3} = w_1$

$z_2 \to \frac{2}{2i-1} = -\frac{2}{5} + i\frac{4}{5} = w_2$

$z_3 \to \frac{2}{2-1} = 2 = w_3$



$\Rightarrow$ it is a circle.

Let us check that $w_1, w_2, w_3 \in \{w \in \mathbb{C} : |w-\frac{2}{3}| = \frac{4}{3}\}$

$|w_1 - \frac{2}{3}| = |-\frac{2}{3} - \frac{2}{3}| = \frac{4}{3}$ $\qquad |w_3 - \frac{2}{3}| = |2 - \frac{2}{3}| = \frac{4}{3}$

$|w_2 - \frac{2}{3}| = |-\frac{2}{5} + i\frac{4}{5} - \frac{2}{3}| = |-\frac{16}{15} + i\frac{12}{15}| = \frac{\sqrt{16^2+12^2}}{15} = \frac{20}{15} = \frac{4}{3}$.

Next, we claim that the circle $|z|=1$ is mapped to $\operatorname{Re} w = -1$. Take 3 points on the boundary: $z_1 = -i$, $z_2 = i$, $z_3 = -1$:

$z_1 \to \frac{2}{-i-1} = -1+i = w_1 \qquad z_2 \to \frac{2}{i-1} = -1-i = w_2$

$z_3 \to \frac{2}{-1-1} = -1 = w_3$

All of them are mapped to $\operatorname{Re} w = -1$. Now let us check what is the image of $|z|<1$. We take a point inside, for instance $z=0$, and obtain $\frac{2}{0-1} = -2$: namely, it is mapped to a point that is outside of $\{w \in \mathbb{C} : |w-\frac{2}{3}| = \frac{4}{3}\} \cap \{w \in \mathbb{C} : \operatorname{Re} w < -1\}$

Now we check what is the image of $\{z \in \mathbb{C} : 1 < |z| < 2\}$. Take $z = \frac{3}{2}$, then $w = \frac{2}{\frac{3}{2}-1} = \frac{2}{\frac{1}{2}} = 4$ – it is in the exterior of $\{w : |w - \frac{2}{3}| = \frac{4}{3}\}$ such that $\operatorname{Re} w > -1$.

d) 1) We are looking for the transformation that maps

$z_1 = 0 \to w_1 = 0$, $\quad z_2 = 1-i \to w_2 = 2-2i = 2(1-i)$, $z_3 = i \to w_3 = 1$

$\Rightarrow$ we obtain the following equations:

1) $\frac{b}{d} = 0$ ($\Rightarrow b=0$) $\qquad$ 2) $\frac{a(1-i)+b}{c(1-i)+d} = 2(1-i)$ $\qquad$ 3) $\frac{ai+b}{ci+d} = 1$

Since (from 1)) $b=0$, we get using 2) and 3):

$$\begin{cases} a(1-i) = 2(1-i)^2 c + 2(1-i)d \\ ai = ci+d \end{cases} \underset{1-i\neq 0}{\Longleftrightarrow} \begin{cases} a = 2c(1-i) + 2d \\ a = c - id \quad (\frac{1}{i} = -i) \end{cases}$$

$\Leftrightarrow c - id = 2c(1-i) + 2d \Leftrightarrow c(2i-1) = d(2+i) \Leftrightarrow d = \frac{c(2i-1)}{2+i}$

and $a = c - \frac{ic(2i-1)}{2+i} = c + \frac{c(2+i)}{2+i} = 2c$

Thus, plugging the values, we get

$$w = \frac{2cz}{cz + \frac{c(2i-1)}{2+i}} \underset{c\neq 0,\text{ thus it cancels}}{=} \frac{2z}{z+\frac{2i-1}{2+i}}$$

let us compute: $\frac{2i-1}{2+i} = \frac{2i-1}{2+i}\cdot\frac{2-i}{2-i} = \frac{5i}{5} = i \Rightarrow w = \frac{2z}{z+i}$

2) $z_1 = i \to w_1 = 0$ 2) $z_2 = -3 \to w_2 = 6+2i$ 3) $z_3 = \infty \to w_3 = 2i$

First: Recall that for $z=\infty$ corresponds $w = \frac{a}{c}$, thus:

1) $\frac{ia+b}{ic+d} = 0$ 2) $\frac{-3a+b}{-3c+d} = 6+2i$ 3) $\frac{a}{c} = 2i$

From 3) : $a = 2ic$. From 1): $ia+b=0 \Rightarrow a = ib$

$\Rightarrow 2ic = ib \Leftrightarrow 2c = b$

From 2): $\frac{-6ic+2c}{-3c+d} = 6+2i \underset{c\neq 0}{\Leftrightarrow} \frac{2-6i}{\frac{d}{c}-3} = 6+2i \Leftrightarrow \frac{d}{c} - 3 = \frac{2-6i}{6+2i}$

Since (CHECK!) $\frac{2-6i}{6+2i} = -i$ we get $d = c(3-i)$

We plug the values and get

$$w = \frac{2icz + 2c}{cz + c(3-i)} \underset{c\neq 0}{=} \frac{2iz+2}{z+(3-i)}$$

e) We have $w_1 = \frac{a_1 z + b_1}{c_1 z + d_1}$, $w_2 = \frac{a_2 z + b_2}{c_2 z + d_2}$, thus:

$$(w_1 \circ w_2)(z) = w_1(w_2(z)) = w_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} =$$

$$= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$$

$\Rightarrow$ The composition is a Möbius transformation with the coefficients given by:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

No, $w_1 \circ w_2 \neq w_2 \circ w_1$ in general. Example: $w_1 = \frac{z}{z+1} \Rightarrow a_1 = 1$, $b_1 = 0$, $c_1 = 1$, $d_1 = 1$; the corresponding matrix is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Let $w_2 = \frac{z+2}{3z+4} \Rightarrow a_2 = 1$, $b_2 = 2$, $c_2 = 3$, $d_2 = 4$, and the corresponding matrix is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The coefficients of $w_1 \circ w_2$ are given by: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$

The coefficients of $W_2 \circ W_1$ are given by

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix}$$

$\Rightarrow \quad W_1 \circ W_2 \neq W_2 \circ W_1.$

2) a) 1) $f(z) = \dfrac{4z+1}{3z^2+5z-2} = \dfrac{1}{2+z} - \dfrac{1}{1-3z}$ (CHECK!)

Using the geometric series we obtain:

$$\frac{1}{2+z} = \frac{1}{2}\cdot\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\cdot\frac{1}{1-(-\frac{z}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

The radius of convergence is $R=2$. In the same way:

$$\frac{1}{1-3z} = \sum_{n=0}^{\infty}(3z)^n = \sum_{n=0}^{\infty} 3^n z^n \Rightarrow R = \frac{1}{3}$$

From uniqueness of the power series:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}3^n z^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} - 3^n\right]z^n,$$

the radius of convergence $R = \frac{1}{3}$.

2) $f(z) = \dfrac{1-\cos z}{z^2}$

We use the powes series (Maclaurin series - about $z_0=0$)

$$\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$$

and obtain:

$$f(z) = \frac{1-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}}{z^2} = \frac{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)}{z^2} = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \dots$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-2}}{(2n)!}$$

The radius of convergence:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(2n)!} : \frac{(-1)^{n+2}}{(2(n+1))!}\right|$$

$$= \lim_{n\to\infty}|(2n+2)(2n+1)| = \infty$$

b) We are looking for the Taylor series expansion about $z_0 = \pi i$ for $f(z) = e^z$, namely for the series of the form $\sum_{n=0}^{\infty} a_n(z-\pi i)^n$ let us compute the value of the derivatives of f at $z_0 = \pi i$:

$f^{(0)}(z) = f(z)$: $f(\pi i) = e^{i\pi} = \cos\pi + i\sin\pi = -1$

Since for any $k \geq 1$ $f^{(k)}(z) = f(z) = f^{(0)}(z)$, we get

$f^{(k)}(\pi i) = e^{\pi i} = -1$ for all $k \ge 0$. Thus:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} \frac{-1}{n!}(z-\pi i)^n = -\sum_{n=0}^{\infty} \frac{(z-\pi i)^n}{n!}$$

CHECK: The radius of convergence : $R=\infty$

3) a) We have seen that (about $z_0=0$)

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

Let us plug $\frac{1}{z}$ instead of z and we get:

$$z^2 \sin\frac{1}{z} = z^2\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\cdot\frac{1}{z^5} - \dots\right)$$

$$= z - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}\frac{1}{z^3} - \dots = z + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n-1}}$$

The radius of convergence:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n}{(2n+1)!} : \frac{(2n+3)!}{(-1)^{n+1}}\right| = \lim_{n\to\infty}(2n+3)(2n+2)$$

$= \infty$ (for all $z\in\mathbb{C}$)

b) i) $\frac{z-7}{z^2+z-2} = -\frac{2}{z-1} + \frac{3}{z+2}$. The function $\frac{3}{z+2}$ is analytic in the disc $|z|<2$, therefore there exists Taylor series expansion

$$\frac{3}{z+2} = \frac{3}{2}\cdot\frac{1}{1+\frac{z}{2}} = \frac{3}{2}\cdot\frac{1}{1-(-\frac{z}{2})} = \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

(valid for $|z|<2$)

The function $-\frac{2}{z-1}$ is analytic on the domain $|z|>1$, thus we can expand it in negative powers of z

$$-\frac{2}{z-1} = -\frac{2}{z(1-\frac{1}{z})} = -\frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -2\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} = -2\sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n$$

This expansion is valid for all $|z|>1$.

ii) On $|z|>2$ both functions $\frac{3}{z+2}$ and $-\frac{2}{z-1}$ are analytic.

Expand each in negative powes of z:

$$-\frac{2}{z-1} = -\frac{2}{z}\cdot\frac{1}{1-\frac{1}{z}} = -2\sum_{n=1}^{\infty} z^{-n}$$

$$\frac{3}{2+z} = \frac{3}{z}\cdot\frac{1}{1+\frac{2}{z}} = \frac{3}{z}\cdot\frac{1}{1-(-\frac{2}{z})} = \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n = 3\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}}$$

$$= 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{n-1}}{z^n}$$

Combining both parts we get the (unique) Laurent series expansion for $f(z)$ on $|z|>2$: $f(z) = \frac{z-7}{z^2+z-2} = \sum_{n=1}^{\infty}\left[3(-2)^{n-1} - 2\right] z^{-n}$.

iii) $0<|z-1|<1$: The function $\frac{3}{z+2}$ is analytic in $|z-1|<1$, thus there exists Taylor series expansion in powers of $(z-1)$:

$$\frac{3}{z+2} = \frac{3}{3+(z-1)} = \frac{1}{1+\frac{z-1}{3}} = \frac{1}{1-(-\frac{z-1}{3})} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(z-1)^n$$

The term $-\frac{2}{z-1}$ is the negative part of the Laurent series expansion in $0<|z-1|<1$. Therefore:

$$f(z) = \frac{z-7}{z^2+z-2} = -\frac{2}{z-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(z-1)^n \quad \text{on} \quad 0<|z-1|<1.$$

c) $\frac{1-\cos z}{(z-2\pi)^3}$ on $|z-2\pi|>0$, namely $z_0 = 2\pi$:

Since $\cos z = \cos(z-2\pi)$, we get

$$\frac{1-\cos z}{(z-2\pi)^3} = \frac{1-\cos(z-2\pi)}{(z-2\pi)^3} = \frac{1-\left(1-\frac{(z-2\pi)^2}{2!}+\frac{(z-2\pi)^4}{4!}-\frac{(z-2\pi)^6}{6!}+\dots\right)}{(z-2\pi)^3}$$

$$= \frac{1}{z-2\pi} - \frac{(z-2\pi)}{4!} + \frac{(z-2\pi)^3}{6!} - \dots$$

d) The given function is analytic in $0<|z|<1$: the denominator vanishes at $z=0, \pm i$, all outside of $0<|z|<1$. To obtain the series we divide the numerator by the denominator according to the rules of division of polynomials:

$$\frac{2+3z}{z^2+z^4} = \frac{2+3z}{z^2}\cdot\frac{1}{1+z^2} = \left(\frac{2}{z^2}+\frac{3}{z}\right)\frac{1}{1-(-z^2)} \underset{\text{geometric series}}{=} \left(\frac{2}{z^2}+\frac{3}{z}\right)\sum_{n=0}^{\infty}(-z^2)^n$$

$$= \left(\frac{2}{z^2}+\frac{3}{z}\right)\sum_{n=0}^{\infty}(-1)^n z^{2n} = \frac{2}{z^2}+\frac{3}{z}-2-3z+2z^2+3z^3-\dots$$

e) The function $\sqrt{z^2-3z+2}$ cannot be expanded into series on the annulus $1<|z|<2$ since it is not analytic there. The point $z=1$ is a branch point $\Rightarrow$ the given function is not single-valued on the given domain.

4) a) Start with the geometric series and modify:

For $|z|<1=R$ $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \Rightarrow \sum_{n=0}^{\infty}\left(\frac{z-3i}{7}\right)^n$ for $z_0=3i$, $R=7$

(CHECK!)

b) Finding a series which converges for all $z\in\mathbb{C}$ with $\operatorname{Im} z \leq 4$ but diverges for z with $\operatorname{Im} z>4$ is impossible since power series must converge on a disc

c) $\operatorname{Re} z = 8$ is a line, so no power series can be found which only converges on $\operatorname{Re} z = 8$ by same reasoning as part b)

(See Prop 12.1)