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Problem Set 6.

1) a) Show that the image of the disc $|z-1|<2$ under the Möbius transformation $w=\frac{z+1}{z-2}$ is the domain $\{w\in\mathbb{C} \mid |w-2|>2\}$

b) Show that the image of the domain $\{z: \operatorname{Re} z<1\}$ under the Möbius transformation $w=\frac{4z}{z+1}$ is the domain $\{w: |w-3|>1\}$

c) Show that the image of the annulus $\{z: 1<|z|<2\}$ under $w=\frac{2}{z-1}$ is the domain $\{w: \operatorname{Re} w>-1, |w-\frac{2}{3}|>\frac{4}{3}\}$

d) Find the Möbius transformation that maps

i) $0\to 0,\ 1-i\to 2-2i,\ i\to 1$; ii) $i\to 0,\ -3\to 6+2i,\ \infty\to 2i$

e) Let $w_1=\frac{a_1z+b_1}{c_1z+d_1}$, $w_2=\frac{a_2z+b_2}{c_2z+d_2}$ be two Möbius transformations. Prove that the composition $w=w_1\circ w_2$ is also a Möbius transformation and its coefficients can be computed by multiplying $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Is $w_1\circ w_2 = w_2\circ w_1$?

2) a) Find the Maclaurin series expansion for:

1) $f(z)=\frac{4z+1}{3z^2+5z-2}$ 2) $f(z)=\frac{1-\cos z}{z^2}$

What is the radius of convergence?

b) Find the Taylor series expansion about $z_0=\pi i$ for $f(z)=e^z$. What is the radius of convergence?

3) Find the Laurent series expansion for:

a) $z^2\sin\frac{1}{z}$, $0<|z|<R$

b) $\frac{z-7}{z^2+z-2}$ on i) $1<|z|<2$; ii) $|z|>2$ iii) $0<|z-1|<1$

c) $\frac{1-\cos z}{(z-2\pi)^3}$, $|z-2\pi|>0$

d) $\frac{2+3z}{z^2+z^4}$, $0<|z|<1$

e) Let $f(z)=\sqrt{z^2-3z+2}$, $1<|z|<2$. Does the Laurent series expansion for $f(z)$ exist? Justify your answer.

4) Give an example, if possible, of power series with the following properties:

a) centered at $z_0=3i$, with radius of convergence $R=7$

b) centered at $z_0=1$ and convergent for all z with $\operatorname{Im} z\le 4$, but divergent for all z with $\operatorname{Im} z>4$

c) centered at $z_0=0$ and convergent for all z with $\operatorname{Re} z=8$, but divergent for all other $z\in\mathbb{C}$.