# Problem Set 5 - Solutions.

1) a) Root Test Pf: If $L>1$, then there exists $N\in\mathbb{N}$ s.t. $\sqrt[n]{|a_n|}>1$, namely $|a_n|>1$, namely $\{|a_n|\}$ is increasing, thus the series diverges by the Test for Divergence.

Suppose that $L<1$. Choose $r\in\mathbb{R}$ such that $L<r<1$. Since $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=L$ there exists $N\in\mathbb{N}$ s.t. for all $n\geq N$ $0\leq\sqrt[n]{|a_n|}\leq r$, namely, $0\leq|a_n|\leq r^n$. Therefore, for $n\geq N$ the terms of the series $\sum_{n=0}^{\infty}|a_n|$ are bounded by the terms of a convergent geometric series (since $0<r<1$), therefore by the Comparison Test for real series $\sum_{n=0}^{\infty}|a_n|$ converges, namely $\sum_{n=0}^{\infty}a_n$ converges absolutely.

b) α) We use Prop 11.3 1) as follows:

$$\lim_{n\to\infty}\frac{5n-ni}{6n+4}=\lim_{n\to\infty}\frac{5n}{6n+4}-\lim_{n\to\infty}\frac{ni}{6n+4}=\lim_{n\to\infty}\frac{5n}{6n+4}-i\lim_{n\to\infty}\frac{n}{6n+4}$$

$$=\lim_{n\to\infty}\frac{5}{6+\frac{4}{n}}-i\lim_{n\to\infty}\frac{1}{6+\frac{4}{n}}\underset{\lim\limits_{n\to\infty}\frac{a}{n}=0\text{ for any }a}{=}\frac{5}{6}-i\frac{1}{6}$$

β) First, we represent $\frac{3i-4n}{8+2in}$ in Cartesian form $x+iy$:

$$\frac{3i-4n}{8+2in}=\frac{3i-4n}{8+2in}\cdot\frac{8-2in}{8-2in}=\frac{24i-32n+6n+8in^2}{64+4n^2}$$

$$=-\frac{26n}{64+4n^2}+i\frac{8n^2+24}{64+4n^2}=-\frac{\frac{26}{n}}{4+\frac{64}{n^2}}+i\frac{8+\frac{24}{n^2}}{4+\frac{64}{n^2}}$$

$$\Rightarrow\lim_{n\to\infty}\frac{3i-4n}{8+2in}=\lim_{n\to\infty}\left(-\frac{\frac{26}{n}}{4+\frac{64}{n^2}}\right)+i\lim_{n\to\infty}\frac{8+\frac{24}{n^2}}{4+\frac{64}{n^2}}=0+i\frac{8}{4}=2i$$

γ) $\lim\limits_{n\to\infty}\left(4+\frac{i^n}{n}\right)=\lim\limits_{n\to\infty}4+\lim\limits_{n\to\infty}\frac{i^n}{n}=4+\lim\limits_{n\to\infty}\frac{i^n}{n}$

Let us show that $\lim\limits_{n\to\infty}\frac{i^n}{n}=0$, then $\lim\limits_{n\to\infty}\left(4+\frac{i^n}{n}\right)=4$.

Given $\varepsilon>0$ we need to find $N(\varepsilon)\in\mathbb{N}$ s.t. for every $n>N(\varepsilon)$

$$|z_n-z|=\left|\frac{i^n}{n}-0\right|=\left|\frac{i^n}{n}\right|=\frac{|i|^n}{n}=\frac{1}{n}<\varepsilon$$

Choose $N=\left[\frac{1}{\varepsilon}\right]$ (the integer part of $\frac{1}{\varepsilon}$). Then, for $n>N$ $\left|\frac{i^n}{n}\right|<\varepsilon$, namely $\lim\limits_{n\to\infty}\frac{i^n}{n}=0$.

2) a) Let us apply the Ratio Test: $a_n=\frac{i^n n!}{5^n}$, thus

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{i^{n+1}(n+1)!}{5^{n+1}}\cdot\frac{5^n}{i^n n!}\right|=\lim_{n\to\infty}\left|\frac{i(n+1)}{5}\right|=$$

$$\lim_{n\to\infty} \left|\frac{i}{5}\right|(n+1) = \lim_{n\to\infty} \frac{n+1}{5} = \infty$$

Thus, $L = \infty \Rightarrow$ the series diverges.

b) $\sum_{n=1}^{\infty} \frac{ni^n}{2n+1}$ : Let us show that the series diverges by the Test for divergence. $a_n = \frac{ni^n}{2n+1}$:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

$|i^n| = |i|^n = 1$

Thus $\lim_{n\to\infty} a_n \neq 0$ ( $\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$ – why?)
Therefore, the series diverges by the Test for Divergence.

c) $\sum_{n=1}^{\infty}\left(\frac{n}{4^n} + i\frac{3^n}{4^n}\right) = \sum_{n=1}^{\infty} \frac{n}{4^n} + i\sum_{n=1}^{\infty}\frac{3^n}{n^n}$

Let us apply the Ratio test to $\sum_{n=1}^{\infty} \frac{n}{4^n}$ : $a_n = \frac{n}{4^n}$, thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n}\right| = \lim_{n\to\infty} \frac{1}{4}\cdot\frac{n+1}{n} =$$

$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right) = \frac{1}{4}\left(\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}\right) = \frac{1}{4} < 1 \Rightarrow$ the series converges absolutely.

Now we apply the Root (Cauchy) Test to $\sum_{n=1}^{\infty}\frac{3^n}{n^n}$ : $a_n = \frac{3^n}{n^n} = \left(\frac{3}{n}\right)^n$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{3}{n}\right|^n} = \lim_{n\to\infty} \frac{3}{n} = 0 < 1 \Rightarrow$ the series converges absolutely. By Prop 11.3 2) $\sum_{n=1}^{\infty}\left(\frac{n}{4^n} + i\frac{3^n}{4^n}\right)$ converges absolutely.

3) a) The given series are not over all powers of $z$. Denote $z^2 = w$ and consider the standard power series $\sum_{n=0}^{\infty} \frac{w^n}{(2+i)^n}$. By formula for the radius of convergence:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|, \text{ where } a_n = \frac{1}{(2+i)^n}$$

Thus we obtain (for $w$!)

$$R_w = \lim_{n\to\infty}\left|\frac{1}{(2+i)^n}\cdot\frac{(2+i)^{n+1}}{1}\right| = \lim_{n\to\infty}|2+i| = |2+i| = \sqrt{5}$$

Therefore, the series (in $w$) converges on a disc $|w| < \sqrt{5}$ and the given series converges if $|z^2| = |z|^2 < \sqrt{5} \Rightarrow |z| < \sqrt[4]{5} \Rightarrow R = \sqrt[4]{5}$.

b) $\sum_{n=1}^{\infty}\frac{(z-i+2)^n}{4^n+2n}$ – the series is of the form $\sum_{n=1}^{\infty} a_n(z-z_0)^n$

where $z_0 = i-2$ and $a_n = \frac{1}{4^n+2n}$. Let us find the radius of convergence:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}+2(n+1)}{4^n+2n}\right| = \lim_{n\to\infty}\left|\frac{4(4^n+2n)-8n+2(n+1)}{4^n+2n}\right|$$

$$= \lim_{n\to\infty}\left[4 - \frac{6n}{4^n+2n} + \frac{2}{4^n+2n}\right] = 4 - \lim_{n\to\infty}\frac{6n}{4^n+2n} + \lim_{n\to\infty}\frac{2}{4^n+2n}$$

$$= 4 - \lim_{n\to\infty}\frac{\frac{6n}{4^n}}{1+\frac{2n}{4^n}} + \lim_{n\to\infty}\frac{\frac{2}{4^n}}{1+\frac{2n}{4^n}}$$

Let us show that $\lim\limits_{n\to\infty}\frac{n}{4^n}=0$, then

$$\lim_{n\to\infty}\frac{\frac{6n}{4^n}}{1+\frac{2n}{4^n}}=0 \quad \text{and} \quad \lim_{n\to\infty}\frac{\frac{2}{4^n}}{1+\frac{2n}{4^n}}=0 \quad (\text{since } \lim_{n\to\infty}\frac{2}{4^n}=0)$$

$\Rightarrow \lim \left|\frac{a_n}{a_{n+1}}\right| = 4 \Rightarrow R=4$, namely the series converges for $|z-i+2|<4$.

To show that $\lim\limits_{n\to\infty}\frac{n}{4^n}=0$ we will show that $\sum\limits_{n=1}^{\infty}\frac{n}{4^n}$ converges, thus by Prop 11.1 $\lim\limits_{n\to\infty}\frac{n}{4^n}=0$. We apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{4^{n+1}}\cdot\frac{4^n}{n} = \lim_{n\to\infty}\frac{1}{4}\cdot\frac{n+1}{n} = \frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$= \frac{1}{4} < 1 \Rightarrow$ the series converges $\Rightarrow \lim\limits_{n\to\infty}\frac{n}{4^n}=0$.

c) $\sum\limits_{n=0}^{\infty} i^n z^n$: We apply Cauchy-Hadamard formula:

$$R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|i|^n}} = \lim_{n\to\infty}\frac{1}{|i|} = \frac{1}{|i|} = 1$$

$\Rightarrow$ the series converges absolutely for $|z|<1$.

d) $\sum\limits_{n=1}^{\infty}\frac{n!}{n^n}z^n$ - To find the Radius of Convergence we use the ratio formula and obtain

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$\Rightarrow$ the radius of convergence is $e$, namely the series converges for $|z|<e$.

e) $\sum\limits_{n=1}^{\infty}\frac{n^n}{n!}z^n$: we proceed as in d):

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

$\Rightarrow$ the radius of convergence is $\frac{1}{e}$, namely the series converges (absolutely) for $|z|<\frac{1}{e}$.

60

Note, however: as a scalar series:

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1$, but

$\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{n}\right| = e > 1$

The presence of the variable z affect this: the series $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$ converges absolutely for $|z| < \frac{1}{e}$.

f) $\sum_{n=1}^{\infty} \left(\frac{z}{4in}\right)^n$ : We use Cauchy-Hadamard formula:

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{1}{4in}\right|^n}} = \lim_{n\to\infty} \frac{1}{\left|\frac{1}{4in}\right|} =$$

$= \lim_{n\to\infty} |4in| = \lim_{n\to\infty} 4n = \infty \Rightarrow$ the series converges everywhere.

g) $\sum_{n=1}^{\infty} \left(\frac{1}{n} + in\right)(z+1+i)^n$ : $a_n = \frac{1}{n} + in$, $z_0 = -1-i$

Apply the ratio formula:

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n}+in}{\frac{1}{n+1} + i(n+1)}\right| = \lim_{n\to\infty} \frac{\left|\frac{1}{n}+in\right|}{\left|\frac{1}{n+1}+i(n+1)\right|}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^2}+n^2}}{\sqrt{\frac{1}{(n+1)^2}+(n+1)^2}} = \lim_{n\to\infty} \sqrt{\frac{\frac{1+n^4}{n^2}}{\frac{1+(n+1)^4}{(n+1)^2}}} = \lim_{n\to\infty} \sqrt{\frac{(n+1)^2}{n^2} \cdot \frac{1+n^4}{1+(n+1)^4}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} \cdot \sqrt{\frac{1+\frac{1}{n^4}}{1+\frac{4}{n}+\frac{6}{n^2}+\frac{4}{n^3}+\frac{1}{n^4}}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot$$

$\lim \frac{1}{n^k} = 0$ for any $k \geq 0$

$$\cdot \lim_{n\to\infty} \sqrt{\frac{1+\frac{1}{n^4}}{1+\frac{4}{n}+\frac{6}{n^2}+\frac{4}{n^3}+\frac{1}{n^4}}} = 1 \cdot 1 = 1$$

$\Rightarrow$ the radius of convergence is 1 and the series converges absolutely for all $|z+1+i| < 1$.

h) $\sum_{n=0}^{\infty} (z+5i)^{2n}(n+1)^2$ : As in 3 a) define $w = (z+5i)^2$ and consider the series $\sum_{n=0}^{\infty} w^n (n+1)^2$. Then: $a_n = (n+1)^2$,

$$R_w = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^2 =$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right)^2 = \lim_{n\to\infty} \left(1-\frac{1}{n+2}\right) \lim_{n\to\infty} \left(1-\frac{1}{n+2}\right) = 1 \cdot 1 = 1$$

$\lim_{n\to\infty} \frac{1}{n+2} = 0$

$\frac{n+1}{n+2} = \frac{n+2-1}{n+2} = \frac{n+2}{n+2} - \frac{1}{n+2} = 1 - \frac{1}{n+2}$

Therefore, the series in w converges on a disc $|w| < 1$ and the given series converges if $|z+5i|^2<1$, namely $|z+5i|<1$.

4) a) $f(z)=\frac{1}{\alpha z+\beta}=\frac{1}{\beta+\alpha z}\underset{\text{if }\beta\neq 0}{=}\frac{1}{\beta}\cdot\frac{1}{1+\frac{\alpha}{\beta}z}=\frac{1}{\beta}\cdot\frac{1}{1-(-\frac{\alpha}{\beta}z)}$

$\underset{\text{if }|-\frac{\alpha}{\beta}z|<1}{=}\frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\alpha}{\beta}z\right)^n$

Therefore, for $|z|<\left|\frac{\beta}{\alpha}\right|$ we get: $f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{\alpha^n}{\beta^{n+1}}z^n$

If $\beta=0$:

$\frac{1}{\alpha z}=\frac{1}{1-(1-\alpha z)}=\sum_{n=0}^{\infty}(1-\alpha z)^n$ for $|1-\alpha z|<1$.

b) $\sum_{n=0}^{\infty}\frac{1}{12+i}\left(\frac{i}{8}\right)^n$ : Note that $\sum_{n=0}^{\infty}\left(\frac{i}{8}\right)^n$ is a convergent geometric series with $|z|=\left|\frac{i}{8}\right|=\frac{1}{8}<1$. Therefore, by Prop 11.4 2):

Since $\sum_{n=0}^{\infty}\left(\frac{i}{8}\right)^n=\frac{1}{1-\frac{i}{8}}=\frac{1}{1-\frac{i}{8}}\cdot\frac{1+i/8}{1+i/8}=\frac{1+\frac{i}{8}}{1+\frac{1}{64}}=\frac{64(1+\frac{i}{8})}{65}$

$=\frac{64}{65}+i\frac{8}{65}$

$\sum_{n=0}^{\infty}\frac{1}{12+i}\left(\frac{i}{8}\right)^n=\frac{1}{12+i}\left(\frac{64}{65}+i\frac{8}{65}\right)=\frac{1}{12+i}\cdot\frac{12-i}{12-i}\left(\frac{64}{65}+i\frac{8}{65}\right)$

$=\left(\frac{12}{145}-i\frac{1}{145}\right)\left(\frac{64}{65}+i\frac{8}{65}\right)=\frac{768-64i+96i+8}{9425}=$

$=\frac{776}{9425}+i\frac{32}{9425}$

$\Rightarrow$ $\operatorname{Re}S=\frac{776}{9425}$ $\operatorname{Im}S=\frac{32}{9425}$, $|S|=\sqrt{\left(\frac{776}{9425}\right)^2+\left(\frac{32}{9425}\right)^2}$

c) Euler's formula: for any $z\in\mathbb{C}$ $e^{iz}=\cos z+i\sin z$

Pf: By Def 12.4 $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$. Plugging $(iz)$ we obtain:

$e^{iz}=1+iz+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\frac{(iz)^4}{4!}+\frac{(iz)^5}{5!}+\dots=1+iz-\frac{z^2}{2!}-\frac{iz^3}{3!}+\frac{z^4}{4!}+$
$+\frac{iz^5}{5!}+\dots=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots\right)+i\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\right)$

$\underset{\text{Def 12.5}}{=}\cos z+i\sin z$

5) a) $f(z)=\frac{1}{z-z_0}$ is analytic in $\mathbb{C}\setminus\{z_0\}$, but it is not entire - $f(z)$ is not defined at $z_0$.

b) We have seen that $f(z)=|z|^2=x^2+y^2$ is differentiable at $z_0=0$ $(x=y=0)$, but it is not differentiable in any disc centered at $z_0=0$.

Let $f(y) = iy^3$. By CR equations ( $f(x,y) = iy^3 = u(x,y) + iv(x,y)$
$\Rightarrow u(x,y) = 0 \quad v(x,y) = y^3$ )

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} = 3iy^2 \iff y = 0$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x} = 0 \rightarrow \text{always true (for any } x)$$

$\Rightarrow f$ is differentiable only at the points of the form $(x,0)$, namely only on the real axis.

Can you adjust this example to give a function that is differentiable only on the imaginary axis?