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Problem Set 5.

1) a) Copying the beginning of the proof of the Ratio Test, give a proof of the Root Test.

b) Compute the following limits for complex sequences:

i) $\lim_{n\to\infty} \frac{5n-ni}{6n+4}$ ii) $\lim_{n\to\infty} \frac{3i-4n}{8+2in}$ iii) $\lim_{n\to\infty} \left(4+\frac{i^n}{n}\right)$

2) Check convergence / divergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{i^n n!}{5^n}$ b) $\sum_{n=1}^{\infty} \frac{n i^n}{2n+1}$ c) $\sum_{n=1}^{\infty} \left(\frac{n}{4^n} + i\frac{3^n}{n^n}\right)$

3) Find the radius of convergence for each of the following power series

a) $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2+i)^n}$ [Note: only the even powers of z are present. Define $w=z^2$ study the series with w and then conclude about z]

b) $\sum_{n=1}^{\infty} \frac{(z-i+2)^n}{4^n+2n}$ c) $\sum_{n=0}^{\infty} i^n z^n$ d) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$ e) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

f) $\sum_{n=1}^{\infty} \left(\frac{z}{4in}\right)^n$ g) $\sum_{n=1}^{\infty} \left(\frac{1}{n}+in\right)(z+1+i)^n$ h) $\sum_{n=0}^{\infty} (z+5i)^{2n}(n+1)^2$

4) a) Let $\alpha, \beta \in \mathbb{C}$ be constants. Expand $f(z) = \frac{1}{\alpha z+\beta}$ as a geometric series and find its radius of convergence.

b) If S is the sum of the series $\sum_{n=0}^{\infty} \frac{1}{12+i}\left(\frac{i}{8}\right)^n$, what are $\operatorname{Re} S$, $\operatorname{Im} S$, and $|S|$?

c) Prove Euler's formula: For any $z \in \mathbb{C}$ $e^{iz} = \cos z + i\sin z$

5) a) Give an example of a complex function that is analytic on $\mathbb{C}\setminus\{z_0\}$ but is <u>not</u> entire ($z_0 \in \mathbb{C}$ is a constant). Why does your example satisfy the required properties?

b) Give an example of a complex function that is differentiable at 0 but not analytic on <u>any</u> disc centered at 0. Justify your answer.

c) Give an example of a complex function that is differentiable on the real axis but not anywhere else.