MTH5103 Complex Variables

Week 5 Practice Exercies

These exercises are for your daily practice.

1. Prove the following statements:

   (a) If the limit of a sequence exists, then it is unique.

   (b) If the limit of a series exists, then it is unique.

2. Prove that a sequence $\{z_n\}$ converges to z if and only if the *real* sequence $|z_n - z|$ converges to 0.

3. Define $r_N = s - s_N = \sum_{n=N+1}^{\infty} z_n$. Show that $\sum_{n=0}^{\infty} z_n = s$ if and only if $\lim_{N\to\infty} |r_N| = 0$.

4. Prove Proposition 2 from the Week 5 Lecture Notes.

5. Determine the convergence or divergence of the following series

   (a) $\sum_{n=0}^{\infty} \left(\frac{1+i}{3}\right)^n$

   (b) $\sum_{n=0}^{\infty} \left(\frac{5+i}{2}\right)^n$

6. Consider the power series $\sum_{n=0}^{\infty} \left(\frac{5+i}{2}\right)^n z^n$.

   (a) What is the radius of convergence of the power series?

   (b) What is the disc of convergence (i.e., for what values of z does the series converge)?

7. Prove that $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ converges absolutely for all $z \in \mathbb{C}$. Use this fact and Proposition 7 from the Week 5 Lecture Notes to conclude that $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ also converges absolutely for all $z \in \mathbb{C}$.