Week 5. Lecture 13.

We introduce complex sequences and series, discuss their convergence and use them to construct complex power series. Then, we make several concrete connections with complex differentiable functions.

Complex sequences.

Def 10.1: A sequence $\{z_n\}_{n=0}^{\infty}$, $z_n \in \mathbb{C}$, has limit z (as $n\to\infty$) if for every $\varepsilon>0$ there exists $N=N(\varepsilon)\in\mathbb{N}$ such that for all $n>N$ $|z_n-z|<\varepsilon$. Notation: $\lim\limits_{n\to\infty} z_n = z$.

If the limit does not exist (or ∞), we say that the sequence diverges.

In other words, a sequence of complex numbers $\{z_n\}$ converges to the complex number z iff at every neighborhood of z there are almost all but finite number of points from the sequence:

$$\lim_{n\to\infty} z_n = z \iff \forall \varepsilon>0 \ \exists N(\varepsilon)\in\mathbb{N} \text{ s.t. for every } n>N(\varepsilon) \quad |z_n-z|<\varepsilon$$

Claim: If $\lim\limits_{n\to\infty} z_n$ exists, then it is unique.

Pf: Assume $\lim\limits_{n\to\infty} z_n = \tilde{z}_1$ and $\lim\limits_{n\to\infty} z_n = \tilde{z}_2$. Then, for any $\varepsilon>0$ there exists $N_1\in\mathbb{N}$ s.t. for every $n>N_1$ $|z_n-\tilde{z}_1|<\varepsilon/2$. Also there exists $N_2\in\mathbb{N}$ s.t. for every $n>N_2$ $|z_n-\tilde{z}_2|<\varepsilon/2$. Choose $N=\max\{N_1,N_2\}$, then for every $n>N$

$$|z_n-\tilde{z}_1|<\varepsilon/2 \quad \text{and} \quad |z_n-\tilde{z}_2|<\varepsilon/2.$$

Thus

$$|\tilde{z}_1-\tilde{z}_2| = |\tilde{z}_1-z_n+z_n-\tilde{z}_2| \le |\tilde{z}_1-z_n|+|z_n-\tilde{z}_2| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

Since this is true for any $\varepsilon>0$ we conclude that $\tilde{z}_1=\tilde{z}_2$ ∎

Also, it is possible to show that

The complex sequence $\{z_n\}$ converges to $z \iff$ the real sequence $\{|z_n-z|\}_{n=0}^{\infty}$ converges to 0 (CHECK!)

Ex 12.1: Let $z_n = \left(\frac{i}{2}\right)^n$, $n\in\mathbb{N}$. Then:

$$z_0=1,\ z_1=\frac{i}{2},\ z_2=\frac{i^2}{4}=-\frac{1}{4},\ z_3=\frac{i^3}{8}=-\frac{i}{8},\ z_4=\frac{i^4}{16}=\frac{1}{16},\ \dots$$

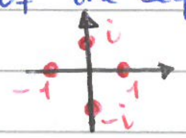
This sequence converges to 0: $\lim\limits_{n\to\infty} z_n = 0$. Can be demonstrated by proving that $|z_n|$ decreases to 0 as n tends to infinity.

CHECK: By graphing the points z_n check that the sequence

$\{z_n\}$ "spirals" in toward the origin (0) in the z-plane.

Ex 12.2: Let $z_n = (2i)^n$: $z_0 = 1, z_1 = 2i, z_2 = -4, z_3 = -8i, \dots$
This sequence diverges: one can show that $\{|z_n|\}_{n=0}^{\infty}$ is unbounded increasing sequence {namely, $|z_n|$ gets arbitrarily large}

Ex 12.3: Let $z_n = (i)^n$: $z_0 = 1, z_1 = i, z_2 = -1, z_3 = -i, z_4 = 1, z_5 = i, \dots$
This is a bounded sequence ($|z_n| \leq 1$ for any n), but $\lim_{n\to\infty} z_n$ does not exist: there is no z such that almost all but finitely many points of the sequence are in a small neighborhood of some z:



z_n jumps at every step.

Important real sequence (limit): let $a_n = (1+\frac{a}{n})^n$ for some a. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+\frac{a}{n})^n = e^a$

Ex 12.4: $\lim_{n\to\infty} (1+\frac{i}{n})^n = e^i$, $\lim_{n\to\infty} (1-\frac{i}{n})^n = \lim_{n\to\infty} (1+\frac{(-i)}{n})^n = e^{-i} = \frac{1}{e^i}$

Prop 10.1: If the sequences $\{z_n\}, \{w_n\}$ of complex numbers converge, then so the sequences $\{z_n \pm w_n\}, \{z_n \cdot w_n\}$, and if $w_n \neq 0$ also $\{\frac{z_n}{w_n}\}$, and

$$\lim_{n\to\infty}(z_n \pm w_n) = \lim_{n\to\infty} z_n \pm \lim_{n\to\infty} w_n; \quad \lim_{n\to\infty} z_n w_n = \lim_{n\to\infty} z_n \lim_{n\to\infty} w_n$$

If $\lim_{n\to\infty} w_n \neq 0$, then: $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{\lim_{n\to\infty} z_n}{\lim_{n\to\infty} w_n}$.

Complex (scalar) series.

Def 11.1: The complex (scalar) series $\sum_{n=1}^{\infty} z_n$ converges to the value $s \in \mathbb{C}$ if the sequence of partial sums $\{S_N\}_{N=0}^{\infty}$, $S_N = z_0 + z_1 + \dots + z_N$ converges to s. The value s is called the sum of the series. If the limit does not exist (or ∞) ($\lim_{N\to\infty} S_N$) then the series diverges.
If the series converges to some value s, then s is unique.

Rmk: Furthermore: if $r_N = s - S_N = \sum_{n=N+1}^{\infty} z_n$, then it is possible to show that $\sum_{n=0}^{\infty} z_n = s \iff \lim_{N\to\infty} r_N = 0$

Namely, the tail of the series converges to 0.

From Def 11.1 we immediately conclude:

Prop 11.1: If $\sum_{n=0}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

Pf: By Def 11.1: If $\sum_{n=0}^{\infty} z_n = S$, then $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} z_n = S$

On the other hand:

$$S_N - S_{N-1} = (z_0 + z_1 + \dots + z_N) - (z_0 + z_1 + \dots + z_{N-1}) = z_N$$

$$\Rightarrow \lim_{N\to\infty} z_N = \lim_{N\to\infty} (S_N - S_{N-1}) \underset{\text{Prop 10.1}}{=} \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} \underset{S \text{ is unique}}{=} S - S = 0 \quad \blacksquare$$

The contrapositive statement is often called the Test for Divergence.

Prop 11.2: (Test for Divergence) Let $\{z_n\}_{n=0}^{\infty} \in \mathbb{C}$. If $\lim_{n\to\infty} z_n \neq 0$ (or does not exist), then the series $\sum_{n=0}^{\infty} z_n$ diverges.

Rmk: The converse of Prop 11.2 is false: $\lim_{n\to\infty} \frac{1}{n} = 0$, but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

As expected there is an immediate relation between convergence or divergence of complex sequences and series to that of real ones.

Prop 11.3: Let $\{z_n\}_{n=0}^{\infty}$, $\sum_{n=0}^{\infty} z_n$ be the complex sequence and the complex series. Suppose that $z = x + iy$ and $S = u + iv$. Then:

1) $\lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ (for $z_n = x_n + iy_n$)

2) $\sum_{n=0}^{\infty} z_n = S \Leftrightarrow \sum_{n=0}^{\infty} x_n = u$ and $\sum_{n=0}^{\infty} y_n = v$

Pf: 1) follows directly from Def 10.1 and the application of triangle inequality: $|x_n - x|, |y_n - y| \le |z_n - z| \le |x_n - x| + |y_n - y|$

2) is proved by decomposing the complex sum into real and imaginary parts $\sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} x_n + i\sum_{n=0}^{\infty} y_n$ and a careful use of 1) [FINISH!]

Hint: If $\sum_{n=0}^{\infty} x_n = u$, $\sum_{n=0}^{\infty} y_n = v \to$ obvious. Assume $\sum_{n=0}^{\infty} z_n = S$, define $X_N = \sum_{n=0}^{N} x_n$, $Y_N = \sum_{n=0}^{N} y_n$, apply 1) to $S_N = \sum_{n=0}^{N} z_n$.

In much of what follows, we can use Prop 11.3 and results in real sequences and series – these are left as an exercise. For example, Prop 10.1 and an analogous proposition for the series:

Prop 11.4: Suppose that $\sum_{n=0}^{\infty} z_n = S$, $\sum_{n=0}^{\infty} w_n = t$ are two convergent series. Let $\alpha \in \mathbb{C}$. Then,

1) $\sum_{n=0}^{\infty} (z_n \pm w_n) = S \pm t$ (converges to $S \pm t$)

2) $\sum_{n=0}^{\infty} \alpha z_n = \alpha s$ (converges to αs)

Pf (EXERCISE — exactly as in the real case - partial sums).

Ex 13.1 (Key example - Geometric series)

Suppose that $|z|<1$.

We claim that the series $\sum_{n=0}^{\infty} z^n$, called the Geometric series, converges.

Claim 11.5: $\sum_{n=0}^{\infty} z^n$ converges (if $|z|<1$) and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

Pf: We prove it using Def 11.1: we will show that the sequence of partial sums converges to $\frac{1}{1-z}$.

Need to show: $\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=0}^{N} z_n = \frac{1}{1-z}$.

First, we obtain a closed formula for S_N as follows.

$$\begin{cases} S_N = 1+z+z^2+\dots+z^N \\ z\cdot S_N = z+z^2+z^3+\dots+z^{N+1} \end{cases} \Rightarrow S_N - zS_N = S_N(1-z) = 1-z^{N+1}$$

$$\Rightarrow S_N = \frac{1-z^{N+1}}{1-z}$$

Now we use this formula to compute the limit

$$\left|\frac{1}{1-z} - S_N\right| = \left|\frac{1}{1-z} - \frac{1-z^{N+1}}{1-z}\right| = \left|\frac{z^{N+1}}{1-z}\right| = \frac{|z|^{N+1}}{|1-z|}$$

Since $|z|<1 \Rightarrow \lim_{N\to\infty} |z|^N = 0$ (from what we know about real sequences)

Therefore

$$\left|\frac{1}{1-z} - S_N\right| \xrightarrow[N\to\infty]{} 0 \Rightarrow \lim_{N\to\infty} S_N = \frac{1}{1-z} \qquad \blacksquare$$

Lectures 14+15.

Now we introduce the notion of absolute convergence.

<u>Def 11.2</u> The series $\sum_{n=0}^{\infty} z_n$ converges absolutely if the series of the moduli of the terms $\sum_{n=0}^{\infty} |z_n|$ converges.

<u>Note</u>: (The Geometric series) For $|z|<1$ $\sum_{n=0}^{\infty} |z|^n$ converges

Namely, the geometric series converges absolutely.

<u>Pf:</u> Follow exactly the same steps as in Ex 13.1, replace z by $|z|$.

The definition of absolute convergence is subtle:

Series can converge, but not absolutely: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

The absolute convergence however always implies convergence:

<u>Prop 11.6</u>: If $\sum_{n=0}^{\infty} |z_n|$ converges, then $\sum_{n=0}^{\infty} z_n$ converges.

<u>Pf:</u> We have $|z_n| = \sqrt{x_n^2+y_n^2} \geq |x_n|, |y_n|$, therefore using the Comparison test for positive (real) series we get that $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge $\Rightarrow$ (by analogous statement for real series) $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge. Thus, by Prop 11.3 2) $\sum_{n=0}^{\infty} z_n$ converges. ■

Now we state 3 very useful tests for convergence of complex series.

<u>Prop 11.7</u>: Let $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$ be a complex series.

(1) <u>Ratio-Test</u>: Consider the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$

- If the limit exists and $L<1$, then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If the limit exists and $L>1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges
- If $L=1$ <u>or</u> the limit does not exist, then the test is inconclusive {namely, there are examples of convergent and divergent series}

(2) <u>Root (Cauchy) Test</u>: Consider the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

- If the limit exists and $L<1$, then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If the limit exists and $L>1$, then the series $\sum_{n=0}^{\infty} a_n$ diverges
- If $L=1$ <u>or</u> does not exist, then the test is inconclusive.

3) Comparison Test:

- If $0 < |a_n| < r_n$ and the positive real series $\sum_{n=0}^{\infty} r_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.
- If $|a_n| > r_n \geq 0$ and $\sum_{n=0}^{\infty} r_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Pf: Ratio Test: If $L > 1$, then the sequence $\{|a_n|\}$ is increasing (for sufficiently large n), therefore the series diverges by the Test for Divergence.

Suppose that $L < 1$. Choose a number r such that $L < r < 1$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$: $0 \leq \left|\frac{a_{n+1}}{a_n}\right| \leq r$. Set $a = a_N$. Then we have

$$|a_{N+1}| \leq r|a_N| = |a|r,$$
$$|a_{N+2}| \leq r|a_{N+1}| < |a|r^2,$$

and, in general $|a_{N+k}| \leq |a|r^k$. Therefore for $n \geq N$ the terms of the series $\sum_{n=0}^{\infty} |a_n|$ are bounded by the terms of a convergent geometric series (since $0 < r < 1$), therefore $\sum_{n=0}^{\infty} |a_n|$ converges by the Comparison Test (for real series), namely, $\sum_{n=0}^{\infty} a_n$ converges absolutely. $\blacksquare$

Root Test and Comparison Test: EXERCISE!

Ex 13.2: Check the convergence of the following series:

a) $\sum_{n=0}^{\infty} (2i)^n$ b) $\sum_{n=0}^{\infty} \frac{(1+i)^n}{2^n}$ c) $\sum_{n=0}^{\infty} \frac{(4i)^n}{n!}$

Sol: 1) We use Comparison Test as follows:

$a_n = (2i)^n \Rightarrow |a_n| = |(2i)|^n = (|2||i|)^n = |2|^n = 2^n > \left(\frac{3}{2}\right)^n$

Since $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ diverges (for example, by the Test for Divergence, since $\lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty \neq 0$) by Comparison Test $\sum_{n=0}^{\infty} (2i)^n$ diverges.

2) Let us apply the Root Test: $a_n = \frac{(1+i)^n}{2^n} = \left(\frac{1+i}{2}\right)^n$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(1+i)}{2}\right|^n} = \lim_{n\to\infty} \left|\frac{1+i}{2}\right| = \frac{|1+i|}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < 1$

$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$ converges absolutely.

c) Let us apply the Ratio Test: $a_n = \frac{(4i)^n}{n!}$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(4i)^{n+1}}{(n+1)!} : \frac{(4i)^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{(4i)^{n+1}}{(n+1)!} \cdot \frac{n!}{(4i)^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{4i}{n+1}\right| = \lim_{n\to\infty} \frac{|4i|}{|n+1|} = \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1 \Rightarrow \text{converges}$$

absolutely.

Complex Power Series

Def 12.1: The series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, is called a power series centered at $z = z_0$ (about $z = z_0$)

A power series is an example of complex series where the sequence z_n from Def 11.1 is of the special form $\{a_n (z-z_0)^n\}$ – involves variable z. Thus, the convergence or divergence of a power series may depend on the values of the variable z. If $z_0 = 0$, the power series are of the form $\sum_{n=0}^{\infty} a_n z^n$.

Rmk: From Def 12.1 every power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for at least one point $z = z_0$.

Next theorem tells us that convergence at one point guarantees convergence on some small disc around z_0. We prove it for the case $z_0 = 0$ (for simplicity of notation), the general case $z_0 \neq 0$ is proved in exactly the same way.

Thm 12.1 (Abel) If $\sum_{n=0}^{\infty} a_n z^n$ converges at $z_0 \neq 0$, then it converges absolutely for any z with $|z| < |z_0|$

Namely, if the series converges at a point z_0, then it converges absolutely at every interior point of the disc centered at 0 of radius $|z_0|$

Pf: Since $\sum_{n=0}^{\infty} a_n z_0^n$ converges by Prop 11.1 $\lim_{n\to\infty} a_n z_0^n = 0$, thus $\{a_n z_0^n\}$ is a bounded sequence: there exists $M \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ $|a_n z_0^n| < M$.

Write the modulus of the general term of the series as:

$$|a_n z^n| = \left|a_n z_0^n \cdot \frac{z^n}{z_0^n}\right| = |a_n z_0^n| \left|\frac{z^n}{z_0^n}\right| < M \left|\frac{z^n}{z_0^n}\right|$$

$\Rightarrow$ since $|z| < |z_0|$, namely $\left|\frac{z}{z_0}\right| < 1$, the general term of the positive series $\sum_{n=0}^{\infty} |a_n z^n|$ is smaller than the one of the positive geometric series $\sum_{n=0}^{\infty} M \left|\frac{z}{z_0}\right|^n \Rightarrow$ by Comparison Test

$\left(\sum_{n=0}^{\infty} |a_n z^n|\right)$ converges $\Rightarrow \sum_{n=0}^{\infty} a_n z^n$ converges absolutely. ☒

Cor 12.1: If $\sum_{n=0}^{\infty} a_n z^n$ diverges at $z = z_1$, then it diverges for every z with $|z| > |z_1|$

Def 12.2 If the series $\sum_{n=0}^{\infty} a_n z^n$ converges for all $z \in U \subseteq \mathbb{C}$, then its sum defines a function on U.

Ex 13.1 (continued): $\sum_{n=0}^{\infty} z^n$ – power series centered at $z_0 = 0$, $a_n = 1$ for all n. Converges on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ (converges absolutely!), and equal to the function $f(z) = \frac{1}{1-z}$ on U.

Next proposition tells us where the power series converges. The proof is omitted.

Prop 12.1: Given a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ there exists $0 \le R \le \infty$ {a real non-negative number or else infinity} such that if $|z - z_0| < R$, then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges absolutely and if $|z - z_0| > R$, then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ diverges

The value R is called the radius of convergence of power series.

A very useful formulae for computing the radius of convergence of power series:

- $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ {for $\sum_{n=0}^{\infty} a_n (z-z_0)^n$}

- Cauchy-Hadamard formula: $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}}$

Ex 14.1 Find the radius of convergence of $\sum_{n=0}^{\infty} (n+i) z^n$

Sol: $a_n = n + i \Rightarrow |a_n| = |n+i| = \sqrt{n^2+1}$

Using Cauchy-Hadamard formula and that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ we get: $R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\sqrt{n^2+1}}} = 1$. Namely: the series converges absolutely for $|z| < 1$.

Rmk: The power series with the radius of convergence R on the circle $|z - z_0| = R$ (or $|z| = R$ if $z_0 = 0$) may converge or diverge or converge at some points and diverge at others.

Ex 14.2: 1) The geometric series $\sum_{n=0}^{\infty} z^n$ – the radius of convergence $R = 1$ and the series diverges at any point on the circle $|z| = 1$ (by the Test for Divergence) CHECK!

2) $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$: The radius of convergence $R = 1$ [CHECK!]

and the series converges (absolutely) for any $|z| \leq 1$

3) One can prove: the series $1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \dots + \frac{z^n}{n} + \dots$ with the radius of convergence $R=1$, diverges at $z=1$ and converges at all other points $|z|=1$ (not absolutely!)

Def 12.3: The disc of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is given by $D = \{z \in \mathbb{C} : |z-z_0| < R\}$ and we write $f(z)$ for the sum of the series on this disc.

Prop 12.2: Suppose the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has disc of convergence D. Then,

1) $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic on D {differentiable at every $z \in D$}

2) For all $z \in D$ the series $\sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ is absolutely convergent and has sum $f'(z)$ (the derivative of f at z)

Note: From Prop 12.2: $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is differentiable arbitrarily many times at any z in the disc of convergence of the series (namely, we can define its higher derivatives)

Def 12.4: For any $z \in \mathbb{C}$ we define the exponential function as the sum of the following series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

If z is real, we get the usual expansion of e^x to series

Claim: e^z is well-defined for all $z \in \mathbb{C}$, namely $R = \infty$.

Pf: By Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 = L < 1$$

and $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty} \left|\frac{z}{n+1}\right| = 0 < 1$ for all $z \in \mathbb{C}$

$\Rightarrow \sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely for all $z \in \mathbb{C}$.

Prop 12.3: e^z is entire and $(e^z)' = e^z$

Pf: By Prop 12.2 1) e^z is analytic for every $z \in \mathbb{C} \Rightarrow$ entire (since $R = \infty$) By Prop 12.2 2):

$$(e^z)' = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \quad \square$$

Prop 12.4: For any $z_1, z_2 \in \mathbb{C}$ $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

Pf: By Def 12.4: $e^{z_1} = \sum_{n=0}^{\infty} \frac{z_1^n}{n!}$ $e^{z_2} = \sum_{n=0}^{\infty} \frac{z_2^n}{n!}$

These are absolutely convergent series $\Rightarrow$ we can multiply them:

$e^{z_1+z_2} = 1 + (z_1+z_2) + \frac{1}{2!}(z_1+z_2)^2 + \dots + \frac{1}{n!}(z_1+z_2)^n + \dots =$

$= 1 + (z_1+z_2) + \left(\frac{z_1^2}{2} + z_1 z_2 + \frac{z_2^2}{2}\right) + \dots + \left(\frac{z_1^n}{n!} + \frac{z_1^{n-1}}{(n-1)!} z_2 + \dots\right.$
$\left. + \frac{z_2^n}{n!}\right) + \dots = e^{z_1} e^{z_2}$ $\boxtimes$

Now we can define trigonometric functions as power series:

Def 12.5: For any $z \in \mathbb{C}$, $|z| < \infty$

$$\sin z = \operatorname{Im} e^z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\cos z = \operatorname{Re} e^z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

For real z this definition coincide with Taylor series expansion of $\sin x$ and $\cos x$ about $x=0$.