Problem Set 4 – Solutions.

1) a) $f(x,y) = 3x + y + i(3y - x)$

$\Rightarrow u(x,y) = 3x + y \qquad v(x,y) = 3y - x$

u and v are differentiable everywhere. Let us check where CR equations hold:

$$\frac{\partial u}{\partial x} = 3 \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial x} = -1 \qquad \frac{\partial v}{\partial y} = 3$$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 1$ everywhere, therefore f is analytic for every $z \in \mathbb{C} \Rightarrow f$ is entire.

b) Denote $f(z) = u(x,y) + iv(x,y)$. Then $\overline{f(z)} = u(x,y) - iv(x,y)$

Since $f(z)$ is analytic in a domain $\Omega \Rightarrow u$ and v are differentiable in a domain Ω and CR equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\overline{f(z)}$ is analytic in a domain Ω as well, thus CR equations hold for $\overline{f(z)}$: $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$. Thus, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial v}{\partial x} = 0$$

$\Rightarrow u$ and v are constant functions $\Rightarrow f$ is a constant function.

c) f is analytic and purely imaginary in a domain Ω:

$f(z) = 0 + iv(x,y)$, namely $u(x,y) = 0$.

Since f is analytic CR equations hold:

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad 0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow v$ is a constant function $\Rightarrow f$ is a constant function.

2) a) i) $f(z) = \dfrac{3 - 3\sqrt{z+1}}{1 + \sqrt{z-1}}$: the branch points are of $\sqrt{z+1}$ and of $\sqrt{z-1}$. Since $\sqrt{z+1} = \sqrt{z-(-1)}$ we get $z_0 = -1$ the branch point of $\sqrt{z+1}$. The branch point of $\sqrt{z-1}$ is $z = 1$. The discontinuity points are the zeros of the denominator and we have

$$1 + \sqrt{z-1} = 0 \iff \sqrt{z-1} = -1 \iff \sqrt{z-1} = i^2 \iff z - 1 = i^4 = 1$$

$$\iff z = 2$$

ii) $\sqrt{(z+8)^3 (2z-4)^5} = \sqrt{(z+8)(z+8)^2 (2z-4)(2z-4)^4} =$

$= \sqrt{z+8}\sqrt{(z+8)^2}\sqrt{2z-4}\sqrt{(2z-4)^4} = \sqrt{z+8}\,(\pm(z+8))\sqrt{2z-4}\,(\pm(2z-4)^2)$

{Rmk: $\sqrt{z^2} = \pm z$ for any z} square root has 2 branches

From the last equality we have 2 branch points: $z=-8$ and $z=2$.

iii) $f(z)=\dfrac{z+\sqrt{z-4}}{z-\sqrt{2z+5}}$: There are 2 branch points $z=4$ and $z=-\frac{5}{2}$ since $z=4$ is the branch point of $\sqrt{z-4}$ and $z=-\frac{5}{2}$ is the branch point of $\sqrt{2z+5}$. The discontinuity points are the points for which $z-\sqrt{2z+5}=0 \Leftrightarrow z=\sqrt{2z+5}$ $\Leftrightarrow z^2=2z+5 \Leftrightarrow z^2-2z-5=0 \Leftrightarrow z_1=1+\sqrt{6}$ or $z_2=1-\sqrt{6}$

b) $f(z)=\sqrt{z-1}$ and $f(0)=-i$

$f(0)=\sqrt{-1}=-i$, thus we have

$f(0)=r\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)=1\left(\cos\left(\frac{\pi+2\pi k}{2}\right)+i\sin\left(\frac{\pi+2\pi k}{2}\right)\right)$ $k=0,1$

$\sqrt{-1}$ attains two values: $+i$ or $-i$. If $k=0$, then the value is $\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)=i$

By assumption $f(0)=-i \Rightarrow$ the chosen branch is $k=1$: we cut the real axis along its positive part up to (including) $z=1$ $\{z=1$ is the branch point of $\sqrt{z-1}\}$

Now, that we know on which branch we are, we can compute $f(i)$:

$f(i)=\sqrt{i-1}$: $|i-1|=\sqrt{2}$ $\cos\theta=-\frac{1}{\sqrt{2}}$, $\sin\theta=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{3\pi}{4}$

$\Rightarrow \sqrt{i-1}=2^{1/4}\left(\cos\left(\frac{\frac{3\pi}{4}+2\pi k}{2}\right)+i\sin\left(\frac{\frac{3\pi}{4}+2\pi k}{2}\right)\right)$ $k=0,1$, but we need to take $k=1$.

$\Rightarrow \sqrt{i-1}=2^{1/4}\left(\cos\left(\frac{3\pi/4+2\pi}{2}\right)+i\sin\left(\frac{3\pi/4+2\pi}{2}\right)\right)=$

$=2^{1/4}\left(\cos\left(\frac{3\pi}{8}+\pi\right)+i\sin\left(\frac{3\pi}{8}+\pi\right)\right)$

$=2^{1/4}\left(-\cos\frac{3\pi}{8}-i\sin\frac{3\pi}{8}\right)$

$\Rightarrow \operatorname{Re} f(i)=-2^{1/4}\cos\frac{3\pi}{8}$.

c) i) $\dfrac{z-\sqrt{z^4-4z^2}}{z^2+6z+9}$ let us find the branch points and the discontinuity points.

The branch points are the points for which $z^4-4z^2=0 \Leftrightarrow$ $z^2(z^2-4)=0 \Leftrightarrow z=0$ or $z=2$ or $z=-2$

The discontinuity points are the zeros of the denominator: $0=z^2+6z+9 \Leftrightarrow z=3$. Therefore, since $z=0\in\{z\in\mathbb{C} \mid |z|<2\}$ the function is not analytic in $|z|<2$ and since ~~z_0~~ $3\in\{|z|>2\}$

the function is not analytic in $|z|>2$.

ii) $f(z)=\dfrac{z^2-1}{z-\sqrt{z^2+4}}$: The branch points are $z=\pm 2i$

Since $z^2+4=0 \Leftrightarrow z=\pm 2i$. The discontinuity points:

$z-\sqrt{z^2+4}=0 \Leftrightarrow z=\sqrt{z^2+4} \Leftrightarrow z^2=z^2+4 \Rightarrow$ no discontinuity points. Since $z=\pm 2i$ are on $|z|=2$ and not in $|z|<2$ or $|z|>2$ we conclude that the function is analytic in both domains

iii) $f(z)=\dfrac{\sqrt{z^2+2z+3}}{z^2-4z+3}$. The branch points:

$z^2+2z+3=0 \Leftrightarrow z_1=-1+i\sqrt{2}$ or $z_2=-1-i\sqrt{2}$.

Note: $|z_1|=|z_2|=\sqrt{3}<2$

The discontinuity points: $z^2-4z+3=0 \Leftrightarrow z=1$ or $z=3$

Since $z=3\in\{z\in\mathbb{C} \mid |z|>2\}$ the function is not analytic in $|z|>2$ and since $z_1,z_2\in\{z\in\mathbb{C} \mid |z|<2\}$ the function is not analytic in $|z|<2$.

3) a) i) $u(x,y)=2e^x\cos y$

$$\frac{\partial u}{\partial x}=2e^x\cos y \qquad \frac{\partial u}{\partial y}=-2e^x\sin y$$
$$\frac{\partial^2 u}{\partial x^2}=2e^x\cos y \qquad \frac{\partial^2 u}{\partial y^2}=-2e^x\cos y \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2e^x\cos y-2e^x\cos y=0$$

$\Rightarrow u(x,y)$ is harmonic function. Let us find its harmonic conjugate using CR equations: $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2e^x\cos y$

$\Rightarrow v(x,y)=\int\frac{\partial v}{\partial y}\,dy=\int 2e^x\cos y\,dy=2e^x\sin y+c(x)$

Let us differentiate $v(x,y)$ with respect to x and compare it to $\frac{\partial v}{\partial x}$ obtained from CR equations (to find $c(x)$)

$$\frac{\partial v}{\partial x}=2e^x\sin y+c'(x)=-\frac{\partial u}{\partial y}=2e^x\sin y$$

$\Rightarrow c=\text{const}\in\mathbb{R}$ and we get: $v(x,y)=2e^x\sin y+c$

ii) $u(x,y)=x^2+2x-y^2 \Rightarrow \frac{\partial u}{\partial x}=2x+2 \quad \frac{\partial u}{\partial y}=-2y$

$\Rightarrow \frac{\partial^2 u}{\partial x^2}=2 \quad \frac{\partial^2 u}{\partial y^2}=-2 \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2+(-2)=0 \Rightarrow u$ is harmonic. We find the harmonic conjugate of u in the same way as in i): $\frac{\partial u}{\partial x}=2x+2=\frac{\partial v}{\partial y}$

$\Rightarrow v(x,y)=\int\frac{\partial v}{\partial y}\,dy=\int(2x+2)\,dy=(2x+2)y+c(x)$

$\Rightarrow \frac{\partial v}{\partial x}=2y+c'(x)=-\frac{\partial u}{\partial y}=2y \Rightarrow c(x)$ is a constant. Thus,

$v(x,y) = 2xy + 2y + c$

b) u is harmonic conjugate of v in Ω, therefore CR equations hold and: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

v is harmonic conjugate of u in Ω, thus we also have:

$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

Therefore: $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \Leftrightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$

Also, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} \Leftrightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$

Thus: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Rightarrow u$ is constant and $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow v$ is constant function.

4) a) $\operatorname{Re} f(z) = u(x,y) = x^2 - y^2 + 2x$ and $f(i) = 2i - 1$

Since f is analytic we are looking for the function v for which $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Differentiation of $u(x,y)$ with respect to x gives us: $\frac{\partial u}{\partial x} = 2x + 2 = \frac{\partial v}{\partial y}$

Now we integrate $\frac{\partial v}{\partial y}$ with respect to y to reconstruct $v(x,y)$

$$v(x,y) = \int \frac{\partial v}{\partial y} dy = \int (2x+2)\,dy = 2xy + 2y + c(x).$$

Differentiation of $u(x,y)$ with respect to y gives

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} = -(2y + c'(x))$$

From the last equality $c'(x) = 0$, namely c is some constant.

$$f(x,y) = x^2 - y^2 + 2x + i(2xy + 2y + c)$$

We find the value of c using the value of f at $z = i = 0 + 1i$

$$f(i) = 0^2 - 1^2 + 2\cdot 0 + i(2\cdot 0\cdot 1 + 2\cdot 1 + c) = -1 + i(2 + c) = 2i - 1$$

Thus, $c = 0$ and

$f(x,y) = x^2 - y^2 + 2x + i(2xy + 2y) = x^2 - y^2 + i\,2xy + 2(x + iy) = z^2 + 2z$

b) We have $\operatorname{Re} z = \operatorname{Im} f(z)$, thus

$z = x + iy \quad f(z) = u(x,y) + iv(x,y) \Rightarrow v(x,y) = x \Rightarrow f(z) = u(x,y) + ix$

From CR equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -1$

Therefore $\frac{\partial u}{\partial y} = -1 \quad \frac{\partial u}{\partial x} = 0$. Integration with respect to y gives:

$$u(x,y) = \int \frac{\partial u}{\partial y} dy = \int -1\,dy = -y + c(x)$$

To find $c(x)$ we differentiate $u(x,y)$ with respect to x and compare to $\frac{\partial u}{\partial x}$ obtained from CR equations:

$$\frac{\partial u}{\partial x} = c'(x) = 0 \Rightarrow c \in \mathbb{R} \text{ is a constant}$$

$\Rightarrow f(x,y) = -y + c + ix = i(x - \frac{1}{i}y) + c = i(x + iy) + c = iz + c$

5) In Ex 9.5 we proved that if f is analytic in some domain and real, then f is a constant function. In particular, if f is entire (analytic on $\mathbb{C}$) and real in $\mathbb{C}$, then f is a constant.

a) $f(z) = \cos|z|^4 = \cos\left(\sqrt{x^2+y^2}\right)^4 = \cos\left((x^2+y^2)^2\right) \in \mathbb{R}$.
Since $\cos\left((x^2+y^2)^2\right)$ is real and not a constant function, by Ex 9.5 it is not entire.

b) $f(z) = z^5 - 4iz^3 + (1+8i)z + (17+i\sqrt{2})$ - f is a polynomial, thus by Ex 9.1 it is an entire function

c) $f(z) = z\,\mathrm{Re}\,z \Rightarrow f(x,y) = (x+iy)x = x^2 + iyx$
$\Rightarrow u(x,y) = x^2 \quad v(x,y) = yx$, u and v are differentiable everywhere. Let us check where CR equations hold.

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = x \qquad \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Leftrightarrow \begin{cases} x = 2x \\ 0 = -y \end{cases} \Leftrightarrow x=0, y=0$$
$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = y$$

Thus, f obeys CR equations only at $(0,0)$ and, in particular, f is not an entire function.

d) $f(z) = \mathrm{Re}\,z = x \in \mathbb{R} \rightarrow f$ is real and not constant $\Rightarrow$ by Ex 9.5 f is not an entire function

e) $f(z) = \mathrm{Im}\,z = y \in \mathbb{R} \rightarrow f$ is real and not constant $\Rightarrow$ by Ex 9.5 f is not an entire function.

f) $f(z) = i\,\mathrm{Re}\,z = ix \rightarrow f$ is purely imaginary and not constant $\Rightarrow$ by 1c) it is not an entire function.