Problem Set 4.

1) a) Prove that $f(z) = 3x+y+i(3y-x)$ is analytic in $\mathbb{C}$, namely: entire

b) Prove that if $f(z)$ and $\overline{f(z)}$ are analytic in a domain Ω, then f is a constant function.

c) Prove that if $f(z)$ is analytic and purely imaginary in a domain Ω, then f is a constant function

2) a) Find all the branch points and all the discontinuity points for the following functions:

i) $f(z) = \dfrac{3-3\sqrt{z+1}}{1+\sqrt{z-1}}$ ii) $f(z) = \sqrt{(z+8)^3(2z-4)^5}$

iii) $f(z) = \dfrac{z+\sqrt{z-4}}{z-\sqrt{2z+5}}$

b) compute $\operatorname{Re} f(i)$ when: $f(z) = \sqrt{z-1}$ and $f(0) = -i$

c) Which of the following functions are analytic in $|z| < 2$ and/or $|z| > 2$:

i) $\dfrac{z-\sqrt{z^4-4z^2}}{z^2+6z+9}$ ii) $\dfrac{z^2-1}{z-\sqrt{z^2+4}}$ iii) $\dfrac{\sqrt{z^2+2z+2}}{z^2-4z+3}$

3) a) Prove that the following functions are harmonic and find for each function its harmonic conjugate

i) $2e^x\cos y$ ii) x^2+2x-y^2

b) Prove: If u is harmonic conjugate of v in a domain Ω and v is harmonic conjugate of u in a domain Ω, then u and v are constant functions.

4) a) Find an analytic function (or functions) if

$\operatorname{Re} f(z) = x^2-y^2+2x$ and $f(i) = 2i-1$

b) Find all the analytic functions for which

$\operatorname{Re} z = \operatorname{Im} f(z)$

5) Using Ex 9.5 from Lecture Notes determine which of the following functions are entire. Justify your answer:

i) $\cos|z|^4$

ii) $z^5 - 4iz^3 + (1+8i)z + (17+i\sqrt{2})$

iii) $z \operatorname{Re} z$

iv) $\operatorname{Re} z$

v) $\operatorname{Im} z$

vi) $i\operatorname{Re} z$