MTH5103 Complex Variables

Week 4 Practice Exercies

These exercises are for your daily practice.

1. Derive the Polar form of the Cauchy-Riemann equations: write z as $re^{i\theta}$, but continue to write $f(z) = u(r,\theta) + iv(r,\theta)$. Using a similar method to the proof done in class, deduce
$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}. \tag{1}$$
Conclude that $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$.

2. Give an example (not just a drawing) of a subset $S \subseteq \mathbb{C}$ which is
   (a) open and connected
   (b) open but not connected
   (c) neither open nor connected

3. For each of the open regions you described in the previous problem, given an example of a function which is analytic on precisely that region.

4. State and prove the Mean Value Theorem for a *real* function $f(x)$ of one variable.

5. (Important preparation for next Lecture): Define the following
   (a) The *real* sequence a_n converges/diverges.
   (b) The *real* series $\sum_{n=0}^{\infty} a_n$ converges/diverges.
   (c) The *real* series $\sum_{n=0}^{\infty} a_n$ converges absolutely/converges conditionally.

6. Review some of the following results from your calculus course:
   (a) Under what conditions does the *geometric series* $\sum_{n=0}^{\infty} cr^n$ converge/diverge?
   (b) Use the Test for Divergence to show $\sum_{n=0}^{\infty} \arctan n$ diverges.
   (c) Use the Ratio Test to prove $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges absolutely.
   (d) What is the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-6)^n}{3^n}$?