<u>Note</u> In the various preceding examples (middle-third Cantor set, $\{1,4,9\}$-decimal digit example, checkerboard example) the recursive rule was the <u>same</u> at each step, and there were two <u>scaling factors</u> appearing in each step:

- Side length of 'boxes' $\varepsilon_{n+1} = \alpha \, \varepsilon_n$

  (where $0 < \alpha < 1$)

- Number of 'boxes' $N_{n+1} = \beta \, N_n$

  (where $\beta > 1$ is an integer)

Thus $\varepsilon_n = \varepsilon_0 \, \alpha^n$

$N_n = N_0 \, \beta^n$

This allows a nice formula for the box dimension of the various examples:

$$D = \lim_{\varepsilon \to 0} \frac{\operatorname{Log} N(\varepsilon)}{\operatorname{Log}(1/\varepsilon)}$$

$$= \lim_{n \to \infty} \frac{\operatorname{Log} N_n}{\operatorname{Log}(1/\varepsilon_n)}$$

$$= \lim_{n \to \infty} \frac{\operatorname{Log} N_n}{-\operatorname{Log} \varepsilon_n}$$

$$= \lim_{n \to \infty} \frac{\operatorname{Log}(N_0 \beta^n)}{-\operatorname{Log}(\varepsilon_0 \alpha^n)}$$

$$= \lim_{n \to \infty} \frac{\operatorname{Log} N_0 + \operatorname{Log} \beta^n}{-\operatorname{Log} \varepsilon_0 - \operatorname{Log} \alpha^n}$$

$$= \lim_{n \to \infty} \frac{\operatorname{Log} N_0 + n \operatorname{Log} \beta}{-\operatorname{Log} \varepsilon_0 - n \operatorname{Log} \alpha}$$

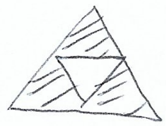
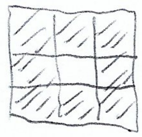
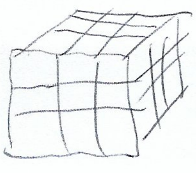
$$= \lim_{n \to \infty} \frac{\frac{1}{n} \operatorname{Log} N_0 + \operatorname{Log} \beta}{-\frac{1}{n} \operatorname{Log} \varepsilon_0 - \operatorname{Log} \alpha}$$

$$= \lim_{n\to\infty} \frac{\log \beta}{-\log \alpha}$$

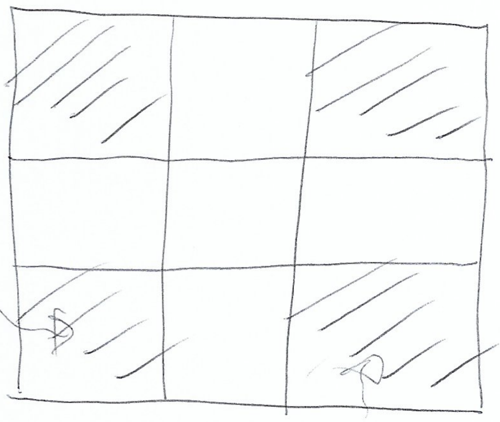
$$= \frac{\log \beta}{-\log \alpha}$$

$$= \frac{\log \beta}{\log(1/\alpha)}$$

We can now give various <u>other</u> box dimensions:

Set	α	β	D
Sierpinski triangle 	$1/2$	3	$\frac{\log 3}{\log 2}$
Sierpinski carpet 	$1/3$	8	$\frac{\log 8}{\log 3}$
Menger sponge 	$1/3$	20	$\frac{\log 20}{\log 3}$ $(\approx 2.7\ldots)$

<u>Example</u> (Similar to checkerboard example)



$$\varphi_1(x,y) = \left(\frac{x}{3}, \frac{y}{3}\right)$$

$$\varphi_2(x,y) = \left(\frac{x+2}{3}, \frac{y}{3}\right)$$

$$\varphi_3(x,y) = \left(\frac{x}{3}, \frac{y+2}{3}\right)$$

$$\varphi_4(x,y) = \left(\frac{x+2}{3}, \frac{y+2}{3}\right)$$

Let $\Phi(A) = \bigcup_{i=1}^{4} \varphi_i(A)$

Let $F_k = \Phi^k([0,1]^2)$

Then $F := \bigcap_{k=0}^{\infty} F_k$ has box dimension

$$D = D(F) = \frac{\log 4}{\log(1/\frac{1}{3})} = \frac{\log 4}{\log 3} \left(= 2 \times \frac{\log 2}{\log 3}\right)$$

<u>Note</u> • $F = C \times C$ where C is the middle-third Cantor set

$2\frac{\log 2}{\log 3}$

$\parallel$

• $D(F) = D(C \times C) = D(C) + D(C) = 2D(C)$

Example: If we also introduce $\cancel{\varphi_5(x,y)}$

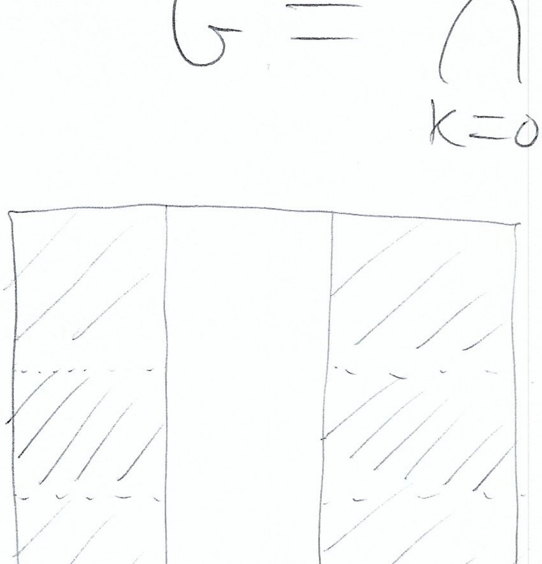
$$\varphi_5(x,y) = \left(\frac{x}{3}, \frac{y+1}{3}\right)$$ ~~and~~ and

$$\varphi_6(x,y) = \left(\frac{x+2}{3}, \frac{y+1}{3}\right),$$ and define

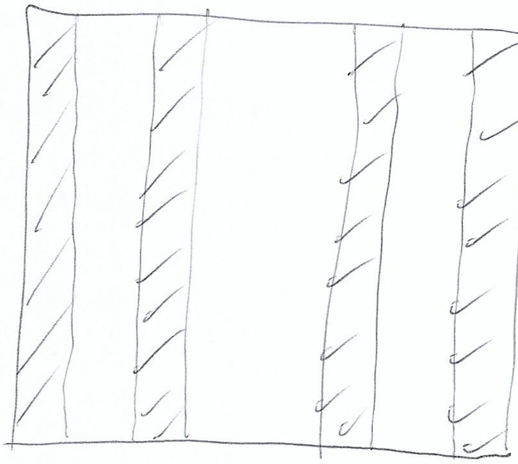
Ψ by $$\Psi(A) = \bigcup_{i=1}^{6} \varphi_i(A),$$

and $$G_k = \Psi^k([0,1]^2),$$

$$G = \bigcap_{k=0}^{\infty} G_k$$



G_1



G_2

Note: $G = C \times [0,1]$, where C is the middle-$\frac{1}{3}$ Cantor set.

The box dimension of G is:

$$D(G) = \frac{\log 6}{\log(1/\frac{1}{3})} = \frac{\log 6}{\log 3}$$

(using the 'scaling factors' formula)

Note that
$$\frac{\log 6}{\log 3} = \frac{\log(2 \times 3)}{\log 3}$$
$$= \frac{\log 2 + \log 3}{\log 3}$$
$$= \frac{\log 2}{\log 3} + 1$$
$$= D(C) + D([0,1]).$$