①

Revision lecture (Revise most of the main topics of the course)

① Modulus, argument, n-th root, n-th power.

$z = x+iy$; $|z| = \sqrt{x^2+y^2}$; the argument of $z \neq 0$: $\arg z$, $z \neq 0$ is an angle θ s.t. $\cos\theta = \frac{x}{|z|}$, $\sin\theta = \frac{y}{|z|}$ (defined up to addition of multiples of 2π)

Ex1: Find polar form of

a) $4-4i$ b) $\frac{2}{1+i\sqrt{3}}$

Sol: a) $|4-4i| = \sqrt{16+16} = 4\sqrt{2}$ $\cos\theta = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{-4}{4\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{7\pi}{4} + 2\pi k \quad k \in \mathbb{Z}$.

$\Rightarrow 4-4i = 4\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \underset{\text{Euler's formula}}{=} 4\sqrt{2}\, e^{i\frac{7\pi}{4}}$

$\boxed{\text{Euler's formula: } e^{i\theta} = \cos\theta + i\sin\theta}$

b) $\frac{2}{1+i\sqrt{3}} = \frac{2(1-i\sqrt{3})}{4} = \frac{1}{2}(1-i\sqrt{3}) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\Rightarrow \left|\frac{2}{1+i\sqrt{3}}\right| = \left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$. $\cos\theta = \frac{\frac{1}{2}}{1} = \frac{1}{2}$, $\sin\theta = \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{5\pi}{3} + 2\pi k, \ k \in \mathbb{Z}$

$\Rightarrow \frac{2}{1+i\sqrt{3}} = 1\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = e^{i\frac{5\pi}{3}}$

From Euler's formula:

$e^{2\pi k i} = \cos 2\pi k + i\sin 2\pi k = 1 \quad \forall k \in \mathbb{Z}$

$e^{\pi i} = \cos\pi + i\sin\pi = -1$.

De Moivre formula: $z^n = |z|^n\left(\cos(n\theta) + i\sin(n\theta)\right)$

Ex2: Find the value of $\left(\frac{2-2i}{2+2i}\right)^{40}$

Sol: $\left|\frac{2-2i}{2+2i}\right| = \left|\frac{1-i}{1+i}\right| = \frac{|1-i|}{|1+i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ $\left[\frac{2-2i}{2+2i} = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{-2i}{2} = -i\right.$

$\Rightarrow \cos\theta = \frac{0}{1} = 0$, $\sin\theta = \frac{-1}{1} = -1 \Rightarrow \theta = \frac{3\pi}{2} + 2\pi k$

$\Rightarrow \left(\frac{2-2i}{2+2i}\right)^{40} = 1^{40}\left(\cos\left(\frac{3\pi}{2}40\right) + i\sin\left(\frac{3\pi}{2}40\right)\right) = \cos 60\pi + i\sin 60\pi$

$= 1$.

Ex 3 Compute $\left| \dfrac{(4-8i)(1-i)^8}{i^7(3+i)^2} \right|$

$(1-i)^8$: $|1-i| = \sqrt{2}$ $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = -\frac{1}{\sqrt{2}}$ $\Rightarrow$ $\theta = \frac{3\pi}{4} + 2\pi k$

$\Rightarrow (1-i)^8 = (\sqrt{2})^8 \left(\cos\left(\frac{3\pi}{4}8\right) + i\sin\left(\frac{3\pi}{4}8\right)\right) = 16(\cos 6\pi + i\sin 6\pi) = 16$

$i^7 = i^2 \cdot i^2 \cdot i^2 \cdot i = (-1)(-1)(-1)i = -i$ (or $|i| = 1$, $\cos\theta = 0$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2} + 2\pi k$

$\Rightarrow i^7 = 1\left(\cos\frac{\pi}{2}7 + i\sin\frac{\pi}{2}7\right) = \cos\left(\frac{\pi}{2} + 3\pi\right) + i\sin\left(\frac{\pi}{2} + 3\pi\right) = -i$)

$(3+i)^2 = 9 - 1 + 6i = 8 + 6i$

$$\left| \frac{(4-8i)16}{-i(8+6i)} \right| = \frac{16|4-8i|}{2|-i||4+3i|} = 8 \cdot \frac{\sqrt{16+64}}{\sqrt{16+9}} = 8\frac{\sqrt{80}}{\sqrt{25}} = \frac{8}{5}4\sqrt{5} = \frac{32}{\sqrt{5}}$$

n-th root: $z = w^n$, $z, w \in \mathbb{C}$

$\arg z_1 z_2 = \arg z_1 + \arg z_2$ $\quad \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$

$\Rightarrow z = w^n \Leftrightarrow |z| = |w|^n$ and $n \arg w = \arg z \pmod{2\pi}$

Ex 4: Compute $\sqrt[9]{i}$

Sol: $|i| = 1$, $\arg i = \frac{\pi}{2} + 2\pi k$ $\left(\cos\theta = 0, \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2} + 2\pi k\right)$

$\Rightarrow w = \sqrt[9]{i} = \sqrt[9]{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}} = \cos\dfrac{\frac{\pi}{2} + 2\pi k}{9} + i\sin\dfrac{\frac{\pi}{2} + 2\pi k}{9}$ $\quad k = 0, 1, .., 8$

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2 Complex functions. $f(z)$: $f(x,y) = u(x,y) + iv(x,y)$

Ex 5: Find Im and Re parts of $f(z) = \frac{z+1}{z-1}$

$z = x + iy$. First, rewrite f in terms of x and y

$$\frac{z+1}{z-1} = \frac{x+iy+1}{x+iy-1} = \frac{(x+1)+iy}{(x-1)+iy} = \frac{((x+1)+iy)((x-1)-iy)}{(x-1)^2+y^2} =$$

$$= \frac{(x+1)(x-1)+y^2+iy(x-1-x-1)}{(x-1)^2+y^2} = \frac{x^2-1+y^2+i(-2y)}{(x-1)^2+y^2} = \frac{x^2+y^2-1}{(x-1)^2+y^2} + i\frac{-2y}{(x-1)^2+y^2}$$

$= u(x,y) + iv(x,y) = \mathrm{Re}\, f + i\,\mathrm{Im}\, f$.

We have also studied how to rewrite back from x, y to z:

$$\mathrm{Re}\, z = x = \frac{1}{2}(z + \bar{z}) \qquad \mathrm{Im}\, z = y = \frac{1}{2i}(z - \bar{z})$$

Ex 6: Rewrite $f(x,y) = x^2 + y^2 - i(x-y)$

$x^2 = \frac{1}{4}(z+\bar{z})^2 = \frac{1}{4}(z^2 + \bar{z}^2 + 2z\bar{z})$; $\quad y^2 = -\frac{1}{4}(z^2 + \bar{z}^2 - 2z\bar{z})$

$ix = \frac{i}{2}(z + \bar{z})$ $\quad iy = \frac{1}{2}(z - \bar{z})$

$$x^2 + y^2 - i(x-y) = x^2 + y^2 - ix + iy = \frac{1}{4}(z^2 + \bar{z}^2 + 2z\bar{z}) - \frac{1}{4}(z^2 + \bar{z}^2 - 2z\bar{z})$$

$$-\frac{i}{2}(z + \bar{z}) + \frac{1}{2}(z - \bar{z}) = \underbrace{z\bar{z}}_{=|z|^2\,(!)} + \frac{1}{2}z(1-i) - \frac{1}{2}\bar{z}(1+i)$$

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③ Transformations.

Ex 7: Find the image of $x=3$ under

a) z^2 b) $\frac{1}{z}$

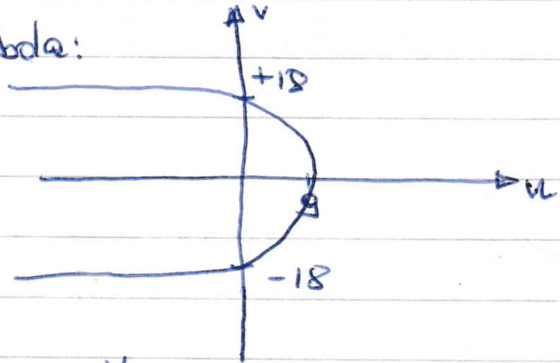
Sol: a) $z^2=(x+iy)^2=x^2-y^2+i(2xy)=u(x,y)+iv(x,y)$

$$\begin{cases} u=x^2-y^2 \\ v=2xy \end{cases} \quad (x\neq 0)\ y=\frac{v}{2x} \Rightarrow u=x^2-\frac{v^2}{4x^2}.$$ For $x=3$ we get

$u=9-\frac{v^2}{36}$ $(\Leftrightarrow)$ $v^2=36(9-u))$

If $v=0 \Rightarrow u=9$; if $u=0$: $\frac{v^2}{36}=9 \Leftrightarrow v^2=9\cdot 36 \Leftrightarrow v=\pm 18$

$\Rightarrow$ the image is the following parabola:

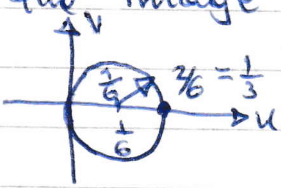


b) $u+iv=\frac{1}{x+iy} \Rightarrow x+iy=\frac{1}{u+iv}=\frac{u}{u^2+v^2}+i\frac{-v}{u^2+v^2} \Rightarrow x=\frac{u}{u^2+v^2},\ y=\frac{-v}{u^2+v^2}$

Now we plug $x=3$ and get: $3=\frac{u}{u^2+v^2} \Leftrightarrow 3u^2-u+3v^2=0$

Now we complete the square in u and get $\left(u-\frac{1}{6}\right)^2+v^2=\frac{1}{36}\left(\frac{1}{6}\right)^2$

$\Rightarrow$ the image is a circle centered at $(\frac{1}{6},0)$ of radius $\frac{1}{6}$:



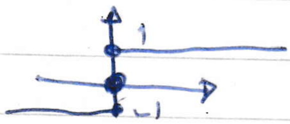
④ Continuity, differentiability, CR eq-s, Branch points, Harmonic f-s

Ex 8: Prove that:

a) $f(z)=\begin{cases} \frac{\operatorname{Re} z}{|z|} & z\neq 0 \\ 0 & z=0 \end{cases}$ is not continuous at $z=0$

b) $f(z)=\begin{cases} \frac{(\operatorname{Re} z)^3}{|z|} & z\neq 0 \\ 0 & z=0 \end{cases}$ is continuous at $z=0$

Sol: a) For $y=0$ we get $f(x,y)=\begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$



therefore it is not continuous at $z=0$ (The lim does not exist!)

Alternatively: $z=re^{i\theta}=r\cos\theta+ir\sin\theta \Rightarrow \operatorname{Re} z=r\cos\theta,\ |z|=r$

Euler's formula

$\lim\limits_{z\to 0}\frac{\operatorname{Re} z}{|z|}=\lim\limits_{r\to 0}\frac{r\cos\theta}{r}=\lim\limits_{r\to 0}\cos\theta=\cos\theta$ – the limit takes different values for different θ $\Rightarrow$ does not exist.

b) Need to show $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{\sqrt{x^2+y^2}} = 0.$ ⊛

Choose $\delta = \varepsilon$. Then, since $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} \underset{\text{cond of continuity}}{\leq} \delta$ $(\Rightarrow |x^2| \leq |x| \leq \delta)$

$\left|\frac{x^3}{\sqrt{x^2+y^2}} - 0\right| \underset{\sqrt{x^2+y^2} \geq \sqrt{x^2}}{\leq} \left|\frac{x^3}{\sqrt{x^2}}\right| = |x^2| \leq \delta = \varepsilon \Rightarrow$ the limit ⊛ exists and $=0$

CR eq-s: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Ex 9: Check whether the following functions are differentiable:

a) $f(x,y) = x^3 - 3xy^2 + i(3x^2y - y^3)$

b) $f(z) = z\,\mathrm{Re}\, z$

Sol: a) $u(x,y) = x^3 - 3xy^2$ $\qquad v(x,y) = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy$ $\qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ everywhere (Not enough! But here: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ defined everywhere and continuous everywhere $\Rightarrow f(x,y)$ is differentiable everywhere!)

Let us compute its derivative. First: rewrite in terms of $z, \bar{z}$

Rewrite in $x = \frac{1}{2}(z+\bar{z})$ $\quad y = \frac{1}{2i}(z-\bar{z})$

$\Rightarrow$ (CHECK!) $f(z) = z^3 \Rightarrow f'(z) = 3z^2$

b) Rewrite in x,y: $z = x+iy \Rightarrow f(x,y) = (x+iy)x = x^2 + iyx$

$u(x,y) = x^2$ $\qquad v(x,y) = yx$

$\frac{\partial u}{\partial x} = 2x$ $\qquad \frac{\partial v}{\partial x} = y$ $\qquad \Rightarrow$ $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right)$ $2x = x \Rightarrow x = 0$

$\frac{\partial u}{\partial y} = 0$ $\qquad \frac{\partial v}{\partial y} = x$ $\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\quad -y = 0 \Rightarrow y = 0$

$\Rightarrow$ differentiable only at $z = 0$!

$f'(0) = 0$ since $\frac{\partial u}{\partial x}\Big|_{(x,y)=(0,0)} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$

Ex 10: Find a,b,c such that $u(x,y) = ax^2 + 2bxy + cy^2$ is harmonic

Sol: u is harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

We have:

$\frac{\partial u}{\partial x} = 2ax + 2by \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2a$ $\qquad \frac{\partial u}{\partial y} = 2bx + 2cy \Rightarrow \frac{\partial^2 u}{\partial y^2} = 2c$

Therefore, we get $2a+2c=0 \iff a=-c$
Hence, u is harmonic if $a=-c$ and b can take any value.

Ex 11: Find the branch points and discontinuity points for
a) $f(z)=\frac{z+1}{z^2-4}$ b) $\sqrt{z^2+3z+2}$

c) $\operatorname{Ln}(z^4-1)$

Sol: a) There are no branch points. The discontinuity points are the zeros of the denominator:
$$z^2-4=0 \iff z^2=4 \iff z=\pm 2$$
b) Branch points: $z^2+3z+2=0 \iff z_{1,2}=\frac{-3\pm\sqrt{9-8}}{2}=-1,-2$
No discontinuity points
c) Branch points: $z^4-1=0 \iff z^4=1 \iff z=\pm 1, \pm i$
(CHECK!)

5 Sequences, Series, Power Series.

Ex 12: Check convergence / divergence of $\sum_{n=1}^{\infty}\frac{(2+i)^n}{2^n}$
We apply the Root (Cauchy) Test:
Let $\sum_{n=1}^{\infty} a_n$, $a_n \in \mathbb{C}$. If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L$ $< 1 \Rightarrow$ converges
$> 1 \Rightarrow$ diverges
$=1$ or does not exist $\Rightarrow$ inconclusive.

Here: $a_n=\frac{(2+i)^n}{2^n} \Rightarrow |a_n|=\left|\frac{2+i}{2}\right|^n=\left(\frac{\sqrt{5}}{2}\right)^n$
$\Rightarrow \lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{\sqrt{5}}{2}=\frac{\sqrt{5}}{2}>1$ $(\sqrt{5}>\sqrt{4}=2) \Rightarrow$ diverges.

Radius of convergence for (power series) $\sum_{n=0}^{\infty} a_n z^n$

1. (Cauchy-Hadamard) $R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}}$

2. $R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

Ex 13: Find the radius of convergence of
a) $\sum_{n=0}^{\infty}(2n+3i)z^n$ b) $\sum_{n=0}^{\infty}\frac{z^{2n}}{(3+4i)^n}$

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a) Compute $|2n+3i| = \sqrt{4n^2+9}$. Now we use Cauchy-Hadamard formula for the radius

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\sqrt{4n^2+9}}} = 1$$

b) This series is not over all powers of z - only even: Denote $w = z^2$, now we get the usual series $\sum_{n=0}^{\infty} \frac{w^n}{(3+4i)^n}$

We use the second formula for the radius

$$R_w = \lim_{n\to\infty} \left| \frac{1}{(3+4i)^n} : \frac{1}{(3+4i)^{n+1}} \right| = |3+4i| = \sqrt{9+16} = 5$$

$\Rightarrow$ the series converges in the disc $|w| < 5$, namely $|z|^2 < 5$
$\Rightarrow |z| < \sqrt{5}$ therefore $R = \sqrt{5}$.

Ex 14: Determine the convergence/divergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

We apply the Ratio Test:

$$a_n = \frac{n!}{n^n} \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} n^n = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{\frac{n+1}{n}}\right)^n$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^n$$

Since $\lim_{n\to\infty} (1+\frac{1}{n})^n = e \Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1 \Rightarrow$ by the Ratio Test the series converges.

⑥ Singularities

Ex 15: Find and classify all the singularities of:

$$a)\ \frac{z - \sin z}{z^6} \qquad b)\ \frac{z^2-1}{z^4+2z^5+z^6}$$

a) We expand $\sin z$ around $z=0$ since the singularity here is at $z=0$. Recall that:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$\Rightarrow \frac{1}{z^6}(z - \sin z) = \frac{1}{z^6}\left(z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots\right) =$$

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$$= \frac{1}{z^6}\left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots\right) = \frac{z^3}{z^6}\left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots\right) =$$

$$= \frac{1}{z^3}\left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots\right)$$

$\Rightarrow$ There is one isolated singularity at $z=0$ and it is a pole of order 3.

b) $$\frac{z^2-1}{z^4+2z^5+z^6} = \frac{z^2-1}{z^4(z^2+2z+1)} = \frac{(z-1)(z+1)}{z^4(z+1)^2} = \frac{z-1}{z^4(z+1)}$$

There are 2 singularities (isolated): $z=0$ and $z=-1$

Look at $\varphi(z) = \frac{z-1}{z+1} \Rightarrow \varphi(0) = -1 \neq 0 \Rightarrow z=0$ is a pole of order 4 (in the exam: formulate the proposition from lectures that is used here)

Now look at $\tilde{\varphi}(z) = \frac{z-1}{z^4}$ $\quad \tilde{\varphi}(-1) = -2 \neq 0 \Rightarrow z=-1$ is a simple pole.

⑦ Residue calculations

Ex 16: Compute the following integrals (all the contours are positively oriented: anticlockwise)

a) $\displaystyle\int_{|z|=5} \frac{z^2+1}{z^2-2z}\,dz$ $\qquad$ b) $\displaystyle\int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z+1)^2}$

Sol: a) 2 singularities: $z=0$ and $z=2$, both inside the contour. Therefore, by Residue Theorem:

$$\int_{|z|=5} \frac{z^2+1}{z^2-2z}\,dz = 2\pi i\left(\operatorname{Res}\left(\frac{z^2+1}{z(z-2)};0\right) + \operatorname{Res}\left(\frac{z^2+1}{z(z-2)};2\right)\right)$$

Now we compute the residue:

$z=0$ is a simple pole since if $\varphi(z) = \frac{z^2+1}{z-2} \Rightarrow \varphi(0) = -\frac{1}{2} \neq 0$

$$\Rightarrow \operatorname{Res}(f;0) = \frac{\varphi(0)}{0!} = -\frac{1}{2}$$

$z=2$ is a simple pole since if $\varphi(z) = \frac{z^2+1}{z} \Rightarrow \varphi(2) = \frac{5}{2} \neq 0$

$$\Rightarrow \operatorname{Res}(f;2) = \frac{\varphi(2)}{0!} = \frac{5}{2}$$

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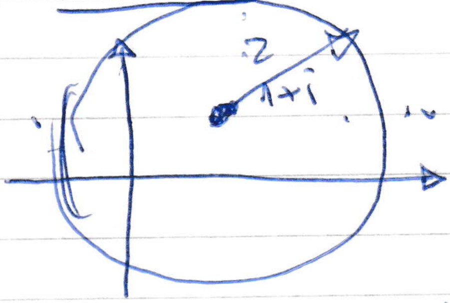
Therefore, we get

$$\int_{|z|=5} \frac{z^2+1}{z(z-2)}\,dz = 2\pi i\left(-\frac{1}{2}+\frac{5}{2}\right) = 4\pi i$$

b) Contour: circle centered at $z=1+i$ of radius 2



2 singularities: $z=1$, $z=-1$

Note: $z=-1$ is outside the contour:

$|-1-1-i| = \sqrt{4+1} = \sqrt{5} > 2$

$\Rightarrow$ the relevant singularity is $z=1$

$\Rightarrow$ by Residue Theorem

$$\int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z+1)^2} = 2\pi i \operatorname{Res}\left(\frac{1}{(z-1)^2(z+1)^2};\, 1\right)$$

$z=1$ is a pole of order 2 (why?)

Let $\varphi(z) = \frac{1}{(z+1)^2}$ $\Rightarrow$ $\operatorname{Res}(f;\,1) = \frac{\varphi'(1)}{1!}$

$\varphi'(z) = -\frac{2}{(z+1)^3}$ $\Rightarrow$ $\varphi'(1) = -\frac{1}{4}$

$$\Rightarrow \int_{|z-1-i|=2} \frac{dz}{(z-1)^2(z+1)^2} = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}.$$